Large Compound Lotteries[*]

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Abstract

Extending preferences over simple lotteries to compound (two-stage) lotteries can be done using two different methods: (1) using the Reduction of compound lotteries axiom, under which probabilities of the two stages are multiplied; (2) using the compound independence axiom, under which each second stage lottery is replaced by its certainty equivalent. Except for expected utility preferences, the rankings induced by the two methods are always in disagreement and deciding on which method to use is not straightforward. Moreover, sometimes each of the two methods may seem to violate some kind of monotonicity. In this paper we demonstrate that, under some conditions, the disagreement disappears in the limit and that for (almost) any pair of compound lotteries, the two methods agree if the second stage lotteries are replicated sufficiently many times.

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1 Introduction

Compound lotteries, that is, lotteries where the outcomes are tickets to simple lotteries, can be reduced to simple lotteries by using the reduction of

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compound lotteries axiom (RCLA), that is, by multiplying the probabilities of the various final outcomes. For example, if $X = (\ldots; x_i, p_i; \ldots)$ and $Y = (\ldots; y_j, q_i; \ldots)$ are lotteries, then the compound lottery $Q = (X, \alpha; Y, 1-\alpha)$ is viewed as $\alpha X + (1-\alpha)Y = (\ldots; x_i, \alpha p_i; y_j, (1-\alpha)q_j; \ldots)$ (see Samuelson [27]). De Finetti's [13] went even further, claiming that probabilities over probabilities are just probabilities, therefore such compound lotteries do not take us out of the original space of lotteries. Denote by $Q_{_R}$ the reduced form of the compound lottery Q using RCLA.

Alternatively, one can use the compound independence axiom (CIA) to reduce compound lotteries recursively, where $Q = (X, \alpha; Y, 1-\alpha)$ is assumed to be indifferent to the simple lottery $(c(X), \alpha; c(Y), 1-\alpha)$ over the certainty equivalents of X and Y. Kreps and Porteus [18] and Segal [28] presented a formal analysis of this procedure. Numerous experiments show this is the way many decision makers view compound lotteries and that RCLA is widely violated. See, e.g., Halevy [16], Chew, Miao, and Zhong [10], Gillen, Snowberg, and Yariv [14], Abdellaoui, Klibanoff, and Placido [1], and Epstein and Halevy [11]. Denote by $Q_{_{CI}}$ the reduced form of Q using CIA.

If preferences are expected utility, then for all Q, $Q_{_R} \sim Q_{_{CI}}$ and expected utility is the only theory to have this property. Moreover, each of the two methods without the other may seem to violate some kind of monotonicity. For example, suppose that X is indifferent to Y, yet $\frac{1}{2}X + \frac{1}{2}Y$ is preferred to both, hence $c(\frac{1}{2}X + \frac{1}{2}Y) > c(X) = c(Y)$. Consider the compound lotteries $Q = (X, \frac{1}{2}; Y, \frac{1}{2})$ and $Q' = (\frac{1}{2}(X-\varepsilon) + \frac{1}{2}Y, 1)$ where $\varepsilon > 0$ is sufficiently small so that $c(\frac{1}{2}(X-\varepsilon) + \frac{1}{2}Y) > c(X) = c(Y)$. Then $(c(\frac{1}{2}(X-\varepsilon) + \frac{1}{2}Y), 1)$ dominates $(c(X), \frac{1}{2}; c(Y), \frac{1}{2})$ by first-order stochastic dominance, yet $\frac{1}{2}X + \frac{1}{2}Y$ first-order stochastically dominates $\frac{1}{2}(X-\varepsilon) + \frac{1}{2}Y$. A decision maker using RCLA will choose Q but will necessarily regret it at the second stage while a decision maker using CIA to choose Q' is getting a lottery that is statistically dominated by an available option. Both seem unsatisfactory. A similar violation occurs if $c(\frac{1}{2}X + \frac{1}{2}Y) < c(X) = c(Y)$, hence in all cases where there are X and Y such that $c(X) = c(Y) \neq c(\frac{1}{2}X + \frac{1}{2}Y)$. It turn out however that this dichotomy disappears when the second stage lotteries are repeated many times. We demonstrate our main argument using the following example.

At age t_2 people find out whether or not they are at risk to suffer from a certain disease later on in life. The ex ante probability of being at risk is q, in which case the person will become sick. There exists a vaccination against this disease: One or two rounds will reduce the conditional probability of getting it (given that a person is at risk). One round will set it at p_1 and two rounds will reduce it to $p_2 < p_1$. The outcome associated with not being sick is $h = 0$ and that of being sick is $s < 0$.[1] Each round will reduce these outcomes by $b > 0$. The last round must be taken by age $t_3 > t_2$ and, as there must be a pause greater than $t_3 - t_2$ between the two rounds, if two rounds are administrated, then the first of the two has to be taken at age $t_1 < t_2$. Consider the following two policies. $\bar{P}$: one round of vaccination at t_3, conditional on being at risk, and $\hat{P}$: two rounds of vaccination, the first at t_1 and conditional on being at risk, the second at t_3.[2] Policies $\bar{P}$ and $\hat{P}$ lead to the compound lotteries $\bar{Q}$ and $\hat{Q}$, respectively (see Fig. 1)

- $\bar{Q} = (0, 1-q;\ X_1, q)$ where $X_1 = (-b, 1-p_1;\ s-b, p_1)$. The reduced form of $\bar{Q}$ is $\bar{Q}_R = (0, 1-q;\ -b, q(1-p_1);\ s-b, qp_1)$

- $\hat{Q} = (-b, 1-q;\ X_2, q)$ where $X_2 = (-2b, 1-p_2;\ s-2b, p_2)$. The reduced form of $\hat{Q}$ is $\hat{Q}_R = (-b, 1-q;\ -2b, q(1-p_2);\ s-2b, qp_2)$

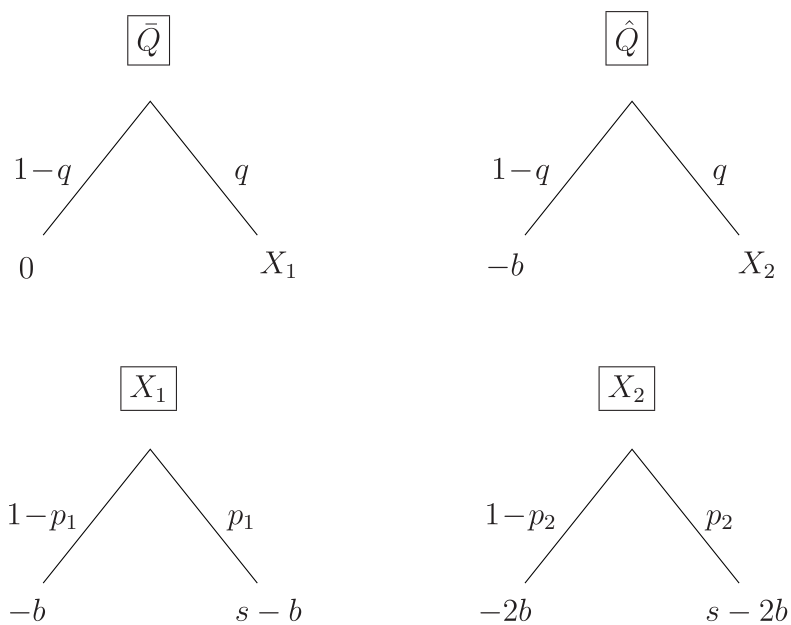

Figure 1: The lotteries Q and X

[1] For simplicity, suppose that all outcomes are monetary payoffs. For example, the disease only effects people's ability to work.

[2] We assume that both policies are better than no vaccination.

Consider a public health official who has to decide which of the two policies to choose for a single individual. The official is using a rank-dependent functional $V(Y) = \int v(t)\mathrm{d}g(F_Y(t))$ (Quiggin [24]), and suppose that $v(x) = x$, $g(p) = 4p$ for $p \leqslant \frac{1}{5}$ and $g(p) = \frac{3+p}{4}$ for $p > \frac{1}{5}$, $p_1 = \frac{2}{5}$, $p_2 = \frac{3}{10}$, $q = \frac{1}{2}$, $b = 1$, and $s = -10$. Then $V(\hat{Q}_R) = -\frac{63}{8} > -\frac{71}{8} = V(\bar{Q}_R)$ yet $V(X_1) = -\frac{38}{4} > -\frac{41}{4} = V(X_2)$. Comparing $\bar{Q}_R$ and $\hat{Q}_R$, decision makers will choose the two-round policy $\hat{P}$. However, diagnosed as not being at risk, the alternative policy yields a greater outcome ($0 > -b$) and, similarly, diagnosed as being at risk, $V(X_1) > V(X_2)$. Comparing $\bar{Q}_{CI}$ and $\hat{Q}_{CI}$, decision makers will choose the one-round policy $\bar{P}$. It is therefore not clear which of the two rules should be used to make such decisions.

Suppose however that there are many identical individuals, and the official has to decide which of the two policies to adopt for all. From his perspective, the outcome of a policy is its total gain or loss. The official is therefore not interested in the utility of each of the individuals, but in the overall sum of their outcomes (recall that in our example all outcomes are monetary payoffs). Our main result (Theorem 1 of section 3) implies that the official's choice does not depend on which of the two rules is used. Moreover, we can tell which of the two policies is better. As a result, we provide officials with a coherent and simple rule of how to choose between such policies. Our analysis is using a technical wrapping assumption which we do not attempt to motivate either on behavioral or normative grounds. However, as we show in section 4, in most commonly used non-expected utility models, violations of this assumption lead to uncompelling behavioral patterns. Section 5 extends our main result to the case of cautious expected utility (Cerreia-Vioglio, Dilleberger, and Ortoleva [4]), even though this model may not satisfy the wrapping assumption. All proofs appear in the appendix.

2 The Model

Let $\mathcal{X}$ be the set of finite real lotteries endowed with the L^1 topology and assume that the decision maker has a complete, transitive, and continuous preference relation $\succeq$ over it. Consider the set of compound lotteries over lotteries $\mathcal{Q} = \{Q = (X_1, q_1; \ldots; X_m, q_m)\}$, where $X_1, \ldots, X_m \in \mathcal{X}$ are of the form $X_i = (x_{i,1}, p_{i,1}; \ldots; x_{i,n_i}, p_{i,n_i})$, $i = 1, \ldots, m$. There are two ways in which compound lotteries can be ranked, and they depend on the way the decision maker transforms two stage lotteries into one stage lotteries (see

Segal [28]).

Reduction of Compound Lotteries Axiom (RCLA) For all $Q \in \mathcal{Q}$,

$$Q \sim Q_{_R} := (\ldots; x_{i,j}, q_i p_{i,j}; \ldots)$$

Compound Independence Axiom (CIA) For all $Q \in \mathcal{Q}$,

$$Q \sim Q_{_{CI}} := (c(X_1), q_1; \ldots; c(X_m), q_m)$$

where $c(X)$, the certainty equivalent of X, is given by $\delta_{c(X)} = (c(X), 1) \sim X$.

Consider now the case in which n replicas of $Q = (X_1, q_1; \ldots; X_m, q_m) \in \mathcal{Q}$ are simultaneously played. Let Q^n be the two stage lottery where the first stage determines for each Q which lottery X_i will be played in the second stage. This is done for each lottery Q independently of the other lotteries. In the second stage, the decision maker is facing the sum of n lotteries, each taken from the set $\{X_1, \ldots, X_m\}$. There are $H := m^n$ (m to the power of n) such possible sequences, denote their sums Y_{nj}, $j = 1, \ldots, H$, with the corresponding probabilities μ_{nj}, which are the product of the corresponding q_i probabilities. Observe that being the sum of simple lotteries, each Y_{nj} is a simple lottery. We thus obtain that

$$Q^n = (Y_{n1}, \mu_{n1}; \ldots; Y_{nH}, \mu_{nH}) \tag{1}$$

The two-stage lottery Q^n yields the lotteries Y_{nj} with probabilities μ_{nj}, $j = 1, \ldots, nH$. For example, let $X_1 = (-1, \frac{1}{2}; 0, \frac{1}{2})$, $X_2 = (-3, \frac{1}{2}; 0, \frac{1}{2})$, $Q = (X_1, \frac{1}{2}; X_2, \frac{1}{2})$, and $n = 2$. The four possible sequences are $Y_{21} = X_1 + X_1 = (-2, \frac{1}{4}; -1, \frac{1}{2}; 0, \frac{1}{4})$, $Y_{22} = X_{23} = X_1 + X_2 = X_2 + X_1 = (-4, \frac{1}{4}; -3, \frac{1}{4}; -1, \frac{1}{4}; 0, \frac{1}{4})$, $Y_{24} = X_2 + X_2 = (-6, \frac{1}{4}; -3, \frac{1}{2}; 0, \frac{1}{4})$, and $Q^2 = (Y_{21}, \frac{1}{4}; \ldots; Y_{24}, \frac{1}{4})$.

The lottery $(Q^n)_{_R}$ is obtained by taking the weighted mixture of these lotteries, that is, $\sum_j \mu_{nj} Y_{nj}$. The lottery $(Q^n)_{_{CI}}$ is obtained by replacing each Y_{nj} with its certainty equivalent. For simplicity, we denote them $Q^n_{_R}$ and $Q^n_{_{CI}}$.

3 Main Result

Our analysis depends on a technical wrapping assumption which we later show to be satisfied by most theories in the literature under conditions that can easily be justified.

Wrapping: A preference relation $\succeq$ satisfies wrapping if it can be represented by a functional V with the following property: There exist $\alpha \geqslant 1$ and $\beta \geqslant 0$ such that for all $Y \in \mathcal{X}$, $\mathrm{E}[v(Y)] \geqslant V(Y) \geqslant \alpha \mathrm{E}[v(Y)] - \beta v(\bar{y})$, where $v(z) := V(\delta_z)$ and $\bar{y}$ is the highest possible outcome in Y.[3]

Consider the set $\mathcal{U}$ of all increasing, twice differentiable, and concave vNM utilities such that $\lim_{x \to -\infty} u'(x) = \infty$ and $\lim_{x \to -\infty} -\frac{u''(x)}{u'(x)}$ exists, is positive, and finite. We restrict attention to concave functions v as in the functional forms discussed in the next section this is a necessary condition for risk aversion, which is a natural requirement in the context of social policy making.[4] For $u \in \mathcal{U}$, let

$$a_u := \lim_{x \to -\infty} -\frac{u''(x)}{u'(x)} \tag{2}$$

and define $\varphi_u(x) := -e^{-a_u x}$.

Consider the set $\tilde{\mathcal{Q}} = \{Q \in \mathcal{Q} : \mathrm{E}(Q_R) \leqslant 0\}$. We restrict attention to compound lotteries with non-positive expected value as our proofs use some results from Safra and Segal [26] where this restriction is assumed. In the context of public health policy this is often the case as the choice is between different ways of reducing potential harm. Let $\bar{Q}$ and $\hat{Q}$ be compound lotteries in $\tilde{\mathcal{Q}}$. The next theorem provides conditions under which, for sufficiently large values of n, the ranking of reduced versions of $\bar{Q}^n$ and $\hat{Q}^n$ does not depend on the reduction procedure.

Theorem 1 Suppose that the preference relation $\succeq$ with the representation V satisfies wrapping and that $v(z) = V(\delta_z)$ is in $\mathcal{U}$. Let $\bar{Q}, \hat{Q} \in \tilde{\mathcal{Q}}$. If $\mathrm{E}[\varphi_v(\bar{Q}_R)] > \mathrm{E}[\varphi_v(\hat{Q}_R)]$, then there exists n^* such that for all $n \geqslant n^*$, $\bar{Q}^n_R \succ \hat{Q}^n_R$ and $\bar{Q}^n_{CI} \succ \hat{Q}^n_{CI}$.

[3] This assumption is obviously satisfied if V is EU with $\alpha = 1$ and $\beta = 0$.

[4] All CARA functions of the form $-ke^{-ax}$ with positive a and k are in $\mathcal{U}$, but $\mathcal{U}$ is much larger than that. For example, if $u \in \mathcal{U}$ then so is $u + P$ where P is a polynomial function. Also, the sum of two CARA functions is in $\mathcal{U}$. Note that in both cases the sum is not CARA.

Theorem 1 assumes that the expected utility (with respect to φ_v) of $\bar{Q}_{_R}$ is different from that of $\hat{Q}_{_R}$. This does not mean that if $\mathrm{E}[\varphi_v(\bar{Q}_{_R})] = \mathrm{E}[\varphi_v(\hat{Q}_{_R})]$, then there exists n^* such that for all $n \geqslant n^*$, $\bar{Q}^n_{_R} \sim \hat{Q}^n_{_R}$ and $\bar{Q}^n_{_{CI}} \sim \hat{Q}^n_{_{CI}}$, or even that $\bar{Q}^n_{_R} \succeq \hat{Q}^n_{_R}$ iff $\bar{Q}^n_{_{CI}} \succeq \hat{Q}^n_{_{CI}}$. The reason is that unless one sequence is increasing and the other decreasing, the fact that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ doesn't imply any specific relation between a_n and b_n (see the proof of the theorem).

On the other hand, consider a compound lottery $Q \in \tilde{\mathcal{Q}}$. The set of lotteries $\bar{Q}_{_R}$ such that $\mathrm{E}[\varphi_v(\bar{Q}_{_R})] \neq \mathrm{E}[\varphi_v(Q_{_R})]$ is open and dense in $\mathcal{X}$. In other words, if $\mathrm{E}[\varphi_v(\bar{Q}_{_R})] \neq \mathrm{E}[\varphi_v(Q_{_R})]$, then this inequality holds for all sufficiently small perturbations of Q and $\bar{Q}$, and if $\mathrm{E}[\varphi_v(\bar{Q}_{_R})] = \mathrm{E}[\varphi_v(Q_{_R})]$, then almost all small perturbations of either Q or $\bar{Q}$ will break this equality.

Given two lotteries $\bar{Q}$ and $\hat{Q}$, Theorem 1 needs to know the shape of v as $x \to -\infty$. But if $\bar{Q}_{_R}$ dominates $\hat{Q}_{_R}$ by first-order stochastic dominance, then regardless of the exact form of v, the expected utility of $\bar{Q}_{_R}$ is higher than that of $\hat{Q}_{_R}$. Therefore, not only is $\bar{Q}^n_{_R}$ preferred to $\hat{Q}^n_{_R}$ for all n, but for a sufficiently large n, $\bar{Q}^n_{_{CI}}$ is also preferred to $\hat{Q}^n_{_{CI}}$. Formally:

Conclusion 1 If $\bar{Q}_{_R}$ first-order stochastically dominates $\hat{Q}_{_R}$, then for every $\succeq$ satisfying the assumptions of Theorem 1 and for sufficiently large n, $\bar{Q}^n_{_R} \succ \hat{Q}^n_{_R}$ and $\bar{Q}^n_{_{CI}} \succ \hat{Q}^n_{_{CI}}$.

4 Functional Forms

We now show conditions under which the wrapping assumption is satisfied by some well-known alternatives to expected utility theory. In all cases, we assume risk aversion. As before, denote the highest outcome of Y by $\bar{y}$.

Rank Dependent Utility

The RDU model (Quiggin [24]) is defined by $V(Y) = \int v(t)\mathrm{d}g(F_Y(t))$ where $g : [0,1] \to [0,1]$ is increasing and onto. We assume that v belongs to $\mathcal{U}$. Note that $v(z) = V(\delta_z)$. In this model, risk aversion with respect to mean preserving spreads is obtained iff v and g are concave (Chew, Karni, and Safra [9]). Under this condition, we show that this model satisfies wrapping iff the probability transformation function g is Lipschitz. Lipschitz is weaker than differentiability and implies weak Gâteaux differentiability (see [9]).[5]

Claim 1 If the preference relation $\succeq$ can be represented by an RDU functional where g is Lipschitz, then it satisfies wrapping.

Violations of the assumption that g is Lipschitz lead to doubtful behavior. Since g is concave, being non-Lipschitz implies that $\lim\limits_{\varepsilon \to 0} \frac{g(\varepsilon)}{\varepsilon} = \infty$. For a given outcome y and increment t, let $s(\varepsilon)$ be the added value that makes the decision maker indifferent between δ_y and $(y-t, \varepsilon; y, 1-2\varepsilon; y+s(\varepsilon), \varepsilon)$. We get

$$v(y-t)g(\varepsilon) + v(y)[g(1-\varepsilon) - g(\varepsilon)] + v(y+s(\varepsilon))[1-g(1-\varepsilon)] = v(y)$$

hence

$$\begin{aligned} \frac{v(y+s(\varepsilon)) - v(y)}{v(y) - v(y-t)} &= \frac{g(\varepsilon)}{1-g(1-\varepsilon)} \\ = & \frac{g(\varepsilon)/\varepsilon}{[1-g(1-\varepsilon)]/\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} \infty \end{aligned} \tag{3}$$

(Observe that since g is concave, $[g(1) - g(1-\varepsilon)]/\varepsilon \leqslant [g(1) - g(0)]/1 = 1$). Eq. (3) implies that even for a minuscule downward change in y, the equiprobable compensation $s(\varepsilon)$ goes to infinity as ε goes to 0, regardless of the size of t. Such behavior seems implausible. Moreover, if v is bounded, no such compensation exists.

Remark: Tversky and Kahneman [29] offered cumulative prospect theory (CPT) as a modification of their original model [17]. CPT uses different probability transformation functions for gains and losses, and a convex utility for losses. The transformation functions suggested by the authors, $p^\gamma/[p^\gamma + (1-p)^\gamma]^{1/\gamma}$ are inconsistent with our wrapping assumption, as their derivatives are not bounded at $p = 0$. However, if transformation with bounded derivatives are permitted, and the utility function from losses is concave, our results for the RDU model extend to CPT as well.

[5] A function is weak Gâteaux differentiable if all directional derivatives exist. It follows from [9, Theorem 1's proof] that weak Gâteaux differentiability is sufficient for the connection between risk aversion and the concavity of g.

Weighted Utility

The WU model (see Chew [7]) is given by

$$V(Y) = \int \frac{w(t)}{\int w(t)\mathrm{d}F_Y(t)} \cdot v(t)\mathrm{d}F_Y(t)$$

where w is continuous and zero is not in its image, hence wlg, $w > 0$. We assume that $V(\delta_z) = v(z)$ belongs to $\mathcal{U}$. Chew [7, eq. (5.2)] showed that $-\frac{w'}{w}$ increases the measure of risk aversion derived from the function v. We therefore assume that $\frac{w'}{w} \leqslant 0$. Under these restrictions, we get the following characterization of the wrapping assumption:

Claim 2 If the preference relation $\succeq$ can be represented by a WU functional with bounded w, then it satisfies wrapping.

Violations of the assumption that w is bounded lead to doubtful behavior. Consider the lottery $Y_t = (t, p; 0, 1-p)$ where wlg $v(0) = 0$. If w is unbounded, then

$$\lim_{t\to-\infty} \frac{V(Y_t)}{v(t)} = \lim_{t\to-\infty} \frac{pw(t)v(t)}{[pw(t) + (1-p)w(0)]\, v(t)} = 1$$

In other words, the certainty equivalent of Y_t is almost t, implying that regardless of the probability of loss p, the decision maker is willing to pay almost the whole potential damage to insure himself against this potential loss. Such behavior seems unlikely.

Disappointment Aversion

The DA model (Gul [15]) is given by

$$V(Y) = \int \gamma(t, b, c(Y))v(t)\mathrm{d}F_Y(t)$$

where

$$\gamma(t, b, c(Y)) = \begin{cases} \frac{1}{1+bF_Y(c(Y))} & \delta_t \succ Y \\ \frac{1+b}{1+bF_Y(c(Y))} & Y \succeq \delta_t \end{cases}$$

Risk aversion in this model holds iff v is concave and $b \geqslant 0$ (see theorem 3 in [15]). We assume further that $V(\delta_z) = v(z)$ belongs to $\mathcal{U}$.

Claim 3 The disappointment aversion model satisfies wrapping.

Quadratic Utility

The general quadratic model is of the form

$$V(Y) = \int\int \psi(t,s)\mathrm{d}F_Y(t)\mathrm{d}F_Y(s)$$

where the function ψ is increasing, symmetric, and unique up to positive affine transformations (see Chew, Epstein, and Segal [8]. See also Machina [19, p. 295]). Quadratic preferences satisfy risk aversion iff $\psi_{11} \leqslant 0$. Observe that $v(z) = V(\delta_z) = \psi(z,z)$.

Claim 4 If the preference relation $\succeq$ can be represented by a quadratic functional such that for all x, y, $\psi(x,x)+\psi(y,y) \geqslant \psi(x,y)+\psi(y,x)$, then it satisfies wrapping.[6]

5 Cautious Expected Utility

Let $\mathcal{W}$ be a set of vNM utilities. According to the Cautious Expected Utility (CU) model (Cerreia-Vioglio, Dillenberger, and Ortoleva [4]),

$$V(Y) = \inf_{u\in\mathcal{W}} c_u(Y) \tag{4}$$

where $u(c_u(Y)) = \mathrm{E}[u(Y)]$. We restrict attention to sets $\mathcal{W}$ in Ω, where $\Omega = \{\mathcal{W} \subset \mathcal{U} : \mathcal{W}$ is finite or the convex hull of a finite set of utilities$\}$. We can therefore replace inf with min in eq. (4).

This functional does not necessarily satisfy the second inequality in the definition of wrapping. Observe first that $v(z) = V(\delta_z) = z$. All functions in $\mathcal{U}$ are concave, hence for every Y, $\mathrm{E}[v(Y)] = \mathrm{E}[Y] \geqslant V(Y)$. But there are no $\alpha \geqslant 1$ and $\beta \geqslant 0$ such that $V(Y) \geqslant \alpha\mathrm{E}[v(Y)] - \beta v(\bar{y})$. Let $\mathcal{W} = \{u\}$ where $u(t) = 1 - e^{-t}$ and let $Y_k = (-k, \frac{1}{k}; 0, 1-\frac{1}{k})$. Denote $c(k) = c_u(Y_k)$ to obtain

$$1 - e^{-c(k)} = \tfrac{1}{k}(1-e^k) \Longrightarrow \lim_{k\to\infty} e^{-c(k)} = \lim_{k\to\infty} 1 - \tfrac{1}{k} + \tfrac{e^k}{k} = \infty$$

Hence $c(k) \to -\infty$ and, by construction, $V(Y_k) \to -\infty$ as well. However, as

$$\alpha\mathrm{E}[v(Y_k)] - \beta v(\bar{y}) = \alpha\mathrm{E}[Y_k] = -\alpha,$$

[6] This last property is implied by, but does not imply, supermodularity of ψ. Machina's example, $V(Y) = \mathrm{E}[u(Y)] + (\mathrm{E}[w(Y)])^2$, satisfies this property (this V is quadratic with $\psi(x,y) = (u(x)+u(y))/2 + w(x)w(y)$).

the inequality $V(Y_k) \geqslant \alpha \mathrm{E}[v(Y_k)] - \beta v(\bar{y})$ fails to hold for k sufficiently large. Theorem 2 below shows that even though the functional of eq. (4) does not satisfy wrapping, the conclusion of Theorem 1 holds for the CU model as well.

Let $\succeq$ be CU where $\mathcal{W} \in \Omega$ is generated by the utility functions $u_1, \ldots, u_\ell$. Let $\bar{a} := \max_j \{a_{u_j}\}$ where $a_{u_j} = \lim_{x \to -\infty} -\frac{u_j''(x)}{u_j'(x)}$, and let $\varphi_{\mathcal{W}}(x) = -e^{-\bar{a}x}$.

Theorem 2 Suppose that the preference relation $\succeq$ is CU where $\mathcal{W} \in \Omega$ and let $\bar{Q}, \hat{Q} \in \tilde{\mathcal{Q}}$. If $\mathrm{E}[\varphi_{\mathcal{W}}(\bar{Q}_R)] > \mathrm{E}[\varphi_{\mathcal{W}}(\hat{Q}_R)]$, then there exists n^* such that for all $n \geqslant n^*$, $\bar{Q}^n_R \succ \hat{Q}^n_R$ and $\bar{Q}^n_{CI} \succ \hat{Q}^n_{CI}$.

Theorem 2 assumes that $\mathcal{W}$ is generated by a finite set of utility functions. But it can be extended to the case where all the generating functions exhibit (weakly) decreasing absolute risk aversion, even if this set of functions is not finite, provided $\bar{a} = \sup\{a_u : u \in \mathcal{W}\} < \infty$.

Another case is Gul's [15] model of disappointment aversion. Cerreia-Vioglio, Dilleberger, and Ortoleva [5] show that it is a CU model where $\mathcal{W}$ is the family of its local utilities. This set is not the convex hull of a finite number or utilities (unless $b = 0$), and as these local utilities are not differentiable, they are not in $\mathcal{U}$, yet this model satisfies the conclusion of Theorem 2 (see Claim 3 above).

6 Dutch Books

Violating any of the two methods for analyzing two-stage lotteries exposes decision makers to Dutch books. De Finetti [12] claims that a decision maker whose preferences violate the basic laws of probability theory is exposed to manipulations that inevitably will lose him money. Markowitz [22] and Raiffa [25] presented arguments against changing preferences while moving down a decision tree.

A careful analysis of these arguments shows that they rely on some further assumptions and therefore may not be prove that individual decision makers must follow both RCLA and CIA (see Machina [20] and McClennen [21] for arguments regarding Markowitz and Raiffa's support of dynamic consistent decision rules and Border and Segal [2, 3] for an analysis of Dutch books involving violations of probability theory). But even if the Dutch books are valid, they can hardly be understood as practical arguments. At best they are theoretical arguments that can be used to persuade a reluctant decision maker to follow expected utility theory. But it is certainly conceivable to imagine willingness to pay a hypothetical price to satisfy an intuitive feelings regarding the proper simplification of a two stage lottery.

The example presented in the introduction is an extended version of the Dutch book argument and seems to weigh against violations of RCLA and CIA performed by the public official. It is hard to justify mathematical mistakes done by such officials, and they will have hard time explaining why they chose an option where all possible outcomes are inferior to an alternative option. Theorems 1 and 2 show that none of these arguments can be raised in a large society. For sufficiently large n, the official's decisions are consistent with both RCLA and CIA.

Appendix: Proofs

For $X \in \mathcal{X}$, let X^n be the sum of n independent replicas of X. This notation is consistent with the notation Q^n, as every $X \in \mathcal{X}$ can be identified with $Q = (X, 1) \in \mathcal{Q}$, and the lottery Q^n is obtained by taking the sum of n repetitions of X. Although for $Q = (X, 1)$, Q^n is formally an element of $\mathcal{Q}$, it can be identified, as above, as an element of $\mathcal{X}$.

Let $Q = (X_1, q_1; \ldots; X_m, q_m)$. Consider $(Q^n)_{_R}$, the one stage lottery derived from Q^n using RCLA and $(Q_{_R})^n$, the sum of n repetitions of $Q_{_R}$. Denote the set of outcomes of $\{X_1, \ldots, X_m\}$ by $\bar{X}$. A typical outcome of $(Q^n)_{_R}$, as well as of $(Q_{_R})^n$, is a sum of n (not necessarily different) outcomes in $\bar{X}$. The probability of a such a sum in $(Q^n)_{_R}$ and in $(Q_{_R})^n$ is the product of the compounded probabilities of each of its summands. Hence we get:

Claim 5 Let $Q = (X_1, q_1; \ldots; X_m, q_m)$. Then $(Q^n)_{_R}$ is the same as $(Q_{_R})^n$. ■

Recall that we denote $Q^n_{_R}$ for $(Q^n)_{_R}$ (and hence for $(Q_{_R})^n$) and $Q^n_{_{CI}}$ for $(Q^n)_{_{CI}}$. mm

Proof of Theorem 1: We prove the theorem for a weaker version of wrapping (hence for a possibly larger family of functionals V):

Wrapping*: The preference relation $\succeq$ satisfies wrapping* if it can be represented by a functional V with the following property: There exist $\alpha \geqslant 1$, $\beta \geqslant$

0, and γ such that for all $Y \in \mathcal{X}$, $\mathrm{E}[v(Y)] \geqslant V(Y) \geqslant \alpha \mathrm{E}[v(Y)] - \beta v(\bar{y}) + \gamma$, where $v(z) := V(\delta_z)$ and $\bar{y}$ is the highest possible outcome in Y.

Note that if the preferences $\succeq$ with V, α, β, γ are as in the definition of wrapping*, then so are $\succeq$ with $V + \zeta, \alpha, \beta, \gamma + \zeta(1 - \alpha + \beta)$ for all ζ. We therefore assume wlg that $v(0) = V(\delta_0) = 0$.

Recall that $a_v = \lim_{x \to -\infty} -\frac{v''(x)}{v'(x)}$, $\varphi_v(x) = -e^{-a_v x}$. Let $Q \in \tilde{\mathcal{Q}}$ and let c satisfy $\varphi_v(c) = \mathrm{E}[\varphi_v(Q_R)]$. We show that

$$\lim_{n \to \infty} \frac{c(Q_R^n)}{n} = \lim_{n \to \infty} \frac{c(Q_{CI}^n)}{n} = c \tag{5}$$

Consider $Y \in \{Y_{nj}\}_{j=1}^H$. Since by wrapping* for all Y, $\mathrm{E}[v(Y)] \geqslant V(Y)$,

$$\begin{aligned} v(c(Q_{CI}^n)) &= V(Q_{CI}^n) \\ &\leqslant \mathrm{E}[v(Q_{CI}^n)] = \textstyle\sum_j \mu_{nj} v(c(Y_{nj})) \\ &= \textstyle\sum_j \mu_{nj} V(Y_{nj}) \leqslant \sum_j \mu_{nj} \mathrm{E}[v(Y_{nj})] \\ &= \mathrm{E}[v(\textstyle\sum_j \mu_{nj} Y_{nj})] \\ &= \mathrm{E}[v(Q_R^n)] = v(c_v(Q_R^n)) \end{aligned} \tag{6}$$

Where $c_v(Q_R^n)$ is the certainty equivalent of Q_R^n using expected utility with v and the last equality signs hold since the expected utility model satisfies RCLA and by Claim 5. It follows that

$$c(Q_{CI}^n) \leqslant c_v(Q_R^n) \tag{7}$$

Similarly to eq. (6),

$$\begin{aligned} v(c(Q_R^n)) &= V(Q_R^n) \\ &\leqslant \mathrm{E}[v(Q_R^n)] = v(c_v(Q_R^n)) \end{aligned}$$

Where the inequality follows again from $\mathrm{E}[v(Y)] \geqslant V(Y)$. Hence

$$c(Q_R^n) \leqslant c_v(Q_R^n) \tag{8}$$

Let $\bar{x}$ be the highest possible outcome in Q_R and note that $n\bar{x}$ is at least as high as the highest outcome of Y and therefore it is also at least as high as $c(Y)$, the certainty equivalent of Y. Let $b := v(\bar{x})$. By the wrapping* assumption there exist $\alpha \geqslant 1$, $\beta \geqslant 0$, and γ such that

$$\begin{aligned} V(Y) &\geqslant \alpha \mathrm{E}[v(Y)] - \beta v(\bar{y}) + \gamma \\ &\geqslant \alpha \mathrm{E}[v(Y)] - \beta v(n\bar{x}) + \gamma \\ &\geqslant \alpha \mathrm{E}[v(Y)] - \beta nb + \gamma \end{aligned} \tag{9}$$

where the last inequality follows from the concavity of v and the fact that $v(0) = 0$. Hence, using inequality (9) and the fact that the highest possible outcome in Q^n_{CI} cannot exceed $n\bar{x}$, we get

$$\begin{aligned} V(Q^n_{CI}) &\geqslant \alpha \mathrm{E}[v(Q^n_{CI})] - \beta nb + \gamma \\ &= \alpha \textstyle\sum_j \mu_{nj} v(c(Y_{nj})) - \beta nb + \gamma \\ &= \alpha \textstyle\sum_j \mu_{nj} V(Y_{nj}) - \beta nb + \gamma \\ &\geqslant \alpha \textstyle\sum_j \mu_{nj} [\alpha \mathrm{E}[v(Y_{nj})] - \beta nb + \gamma] - \beta nb + \gamma \\ &= \alpha^2 \mathrm{E}[v((Q^n)_R)] - (\alpha+1)\beta nb + (\alpha+1)\gamma \\ &= \alpha^2 \mathrm{E}[v(Q^n_R)] - (\alpha+1)\beta nb + (\alpha+1)\gamma \end{aligned}$$

Here too, the last two equality signs hold since the expected utility model satisfies RCLA and by Claim 5. Since $V(Q^n_{CI}) = v(c(Q^n_{CI}))$, we get

$$v(c(Q^n_{CI})) \geqslant \alpha^2 v(c_v(Q^n_R)) - (\alpha+1)\beta nb + (\alpha+1)\gamma \tag{10}$$

Next we show that $\lim_{n\to\infty} \frac{c(Q^n_{CI})}{n}$ exists and is equal to c. By Claim 5 and Lemma 6 of Safra and Segal [26],

$$\lim_{n\to\infty} \frac{c_v(Q^n_R)}{n} = \lim_{n\to\infty} \frac{c_v((Q_R)^n)}{n} = c \tag{11}$$

By inequality (7), $c_v(Q^n_R) - c(Q^n_{CI}) \geqslant 0$. If $c_v(Q^n_R) - c(Q^n_{CI}) > 0$, then by the concavity of v

$$v'(c_v(Q^n_R)) \leqslant \frac{v(c_v(Q^n_R)) - v(c(Q^n_{CI}))}{c_v(Q^n_R) - c(Q^n_{CI})}$$

hence even if $c_v(Q^n_R) - c(Q^n_{CI}) = 0$,

$$c_v(Q^n_R) - c(Q^n_{CI}) \leqslant \frac{v(c_v(Q^n_R)) - v(c(Q^n_{CI}))}{v'(c_v(Q^n_R))}$$

It now follows by inequality (10) that

$$\begin{aligned}
c_v(Q^n_R) - c(Q^n_{CI}) &\leqslant \frac{v(c_v(Q^n_R)) - [\alpha^2 v(c_v(Q^n_R)) - (\alpha+1)\beta nb + (\alpha+1)\gamma]}{v'(c_v(Q^n_R))} \\
&= \frac{-[\alpha^2-1]v(c_v(Q^n_R))}{v'(c_v(Q^n_R))} + \frac{(\alpha+1)\beta nb}{v'(c_v(Q^n_R))} + \frac{(\alpha+1)\gamma}{v'(c_v(Q^n_R))}
\end{aligned}$$

By [26, Lemma 4], $\lim_{n\to\infty} c_v(Q^n_R) = -\infty$, hence since $v \in \mathcal{U}$, $\lim_{n\to\infty} v'(c_v(Q^n_R)) = \infty$ and, using l'Hôpital's rule and eq. (2),

$$\lim_{n\to\infty} \frac{v(c_v(Q^n_R))}{v'(c_v(Q^n_R))} = \lim_{x\to-\infty} \frac{v'(x)}{v''(x)} = -\frac{1}{a_v}$$

Therefore, using eq. (7),

$$\begin{aligned}
0 \leqslant \lim_{n\to\infty} \frac{c_v(Q^n_R) - c(Q^n_{CI})}{n} &\leqslant \lim_{n\to\infty} \frac{\alpha^2-1}{n} \cdot \frac{1}{a_v} \\
&+ \lim_{n\to\infty} \frac{(\alpha+1)\beta b}{v'(c_v(Q^n_R))} + \lim_{n\to\infty} \frac{(\alpha+1)\gamma}{nv'(c_v(Q^n_R))} = 0
\end{aligned}$$

By (11) $\lim_{n\to\infty} \frac{c_v(Q^n_R)}{n} = c$, hence $\lim_{n\to\infty} \frac{c(Q^n_{CI})}{n}$ exists and is equal to c.

We now show that $\lim_{n\to\infty} \frac{c(Q^n_R)}{n}$ exists and is equal to c. Once again, using inequality (9), $V(Q^n_R) \geqslant \alpha \mathrm{E}[v(Q^n_R)] - \beta nb + \gamma$. Since $V(Q^n_R) = v(c(Q^n_R))$, we get

$$v(c(Q^n_R)) \geqslant \alpha v(c_v(Q^n_R)) - \beta nb + \gamma$$

Following similar analysis to the one following inequality (10) and using inequality (8) instead of (7), we obtain $\lim_{n\to\infty} \frac{c(Q^n_R)}{n} = c$. Eq. (5) is thus proved.

Let $\bar{Q}, \hat{Q} \in \tilde{\mathcal{Q}}$, let $\bar{c}$ and $\hat{c}$ satisfy $\varphi_v(\bar{c}) = \mathrm{E}[\varphi_v(\bar{Q}_R)]$ and $\varphi_v(\hat{c}) = \mathrm{E}[\varphi_v(\hat{Q}_R)]$, and assume that $\bar{c} > \hat{c}$. By eq. (5), $\bar{c} = \lim_{n\to\infty} \frac{c(\bar{Q}^n_R)}{n} = \lim_{n\to\infty} \frac{c(\bar{Q}^n_{CI})}{n}$ and $\hat{c} = \lim_{n\to\infty} \frac{c(\hat{Q}^n_R)}{n} = \lim_{n\to\infty} \frac{c(\hat{Q}^n_{CI})}{n}$, hence there is n^* such that for all $n \geqslant n^*$, $\bar{Q}^n_{CI} \succ \hat{Q}^n_{CI}$ and $\bar{Q}^n_R \succ \hat{Q}^n_R$. ■

Proof of Claim 1: The concavity of g implies that for all Y, $\mathrm{E}[v(Y)] \geqslant V(Y)$ (see [9]). Suppose g is Lipschitz. We show that there exist $\alpha \geqslant 1$ and

$\beta \geqslant 0$ such that $V(Y) \geqslant \alpha \mathrm{E}[v(Y)] - \beta v(\bar{y})$. Note that being Lipschitz on $[0,1]$ implies the existence of $k \geqslant 1$ such that for all p, $\mathrm{d}g \leqslant k\mathrm{d}p$.[7] Define $v^*(t) = v(t) - v(\bar{y})$ (note that $v^*(t) \leqslant 0$ for all outcomes of Y). Then,

$$\begin{aligned}
V(Y) &= \int v(t)\mathrm{d}g(F_Y(t)) \\
&= \int [v^*(t) + v(\bar{y})]\mathrm{d}g(F_Y(t)) \\
&= \int v^*(t)\mathrm{d}g(F_Y(t)) + v(\bar{y}) \\
&\geqslant k \int v^*(t)\mathrm{d}F_Y(t) + v(\bar{y}) \\
&= k \int [v(t) - v(\bar{y})]\mathrm{d}F_Y(t) + v(\bar{y}) \\
&= k\mathrm{E}[v(Y)] - (k-1)v(\bar{y})
\end{aligned}$$

The claim is obtained by choosing $\alpha = k$ and $\beta = k - 1$. ■

Weighted Utility

The WU model (see Chew [7]) is given by

$$V(Y) = \int \frac{w(t)}{\int w(t)\mathrm{d}F_Y(t)} \cdot v(t)\mathrm{d}F_Y(t)$$

where w is continuous and zero is not in its image, hence wlg, $w > 0$. We assume that $V(\delta_z) = v(z)$ belongs to $\mathcal{U}$. Chew [7, eq. (5.2)] showed that $-\frac{w'}{w}$ increases the measure of risk aversion derived from the function v. We therefore assume that $\frac{w'}{w} \leqslant 0$. Under these restrictions, we get the following characterization of the wrapping assumption:

Claim 6 If the preference relation $\succeq$ can be represented by a WU functional with bounded w, then it satisfies wrapping.

[7] As $\frac{g(1)-g(0)}{1-0} = 1$, there is no $k < 1$ such that for all p, $\mathrm{d}g \leqslant k\mathrm{d}p$.

Proof of Claim 2: Since $w > 0$, the requirement $\frac{w'}{w} \leqslant 0$ is equivalent to $w' \leqslant 0$. For every s,

$$\frac{\int^s w(t)\mathrm{d}F_Y(t)}{\int w(t)\mathrm{d}F_Y(t)} - \int^s \mathrm{d}F_Y(t) \geqslant 0$$

$$\Longleftrightarrow \quad \int^s \left[w(t) - \int w(t)\mathrm{d}F_Y(t)\right] \mathrm{d}F_Y(t) \geqslant 0 \tag{12}$$

At $s = -\infty$ and at $s = \infty$, the last expression equals 0. Let t^* be such that for $t \leqslant t^*$, $w(t) \geqslant \int w(t)\mathrm{d}F_Y(t)$, and for $t > t^*$, $w(t) < \int w(t)\mathrm{d}F_Y(t)$. For $s \leqslant t^*$, inequality (12) is obviously satisfied. For $s > t^*$ it is satisfied since $\int_s \left[w(t) - \int w(t)\mathrm{d}F_Y(t)\right] \mathrm{d}F_Y(t) \leqslant 0$.

Define $G_Y(s) = \int^s w(t)\mathrm{d}F_Y(t) / \int w(t)\mathrm{d}F_Y(t)$. Since $\int^s \mathrm{d}F_Y(t) = F_Y(s)$, it follows by (12) that F_Y first-order stochastically dominates G_Y, hence $\mathrm{E}[v(Y)] \geqslant V(Y)$.

To show that if w is bounded then there exist $\alpha \geqslant 1$ and $\beta \geqslant 0$ such that $V(Y) \geqslant \alpha\mathrm{E}[v(Y)] - \beta v(\bar{y})$, suppose $\{w(\cdot)\} \subset [a, b]$ where $b > a > 0$. Choose $\alpha > \frac{b}{a}$ and $\beta = \alpha - 1$ and define $v^*(t) = v(t) - v(\bar{y})$ (note that $v^*(t) \leqslant 0$ for all outcomes of Y). Then,

$$\begin{aligned}
V(Y) &= \int \frac{w(t)}{\int w(t)\mathrm{d}F_Y(t)} \cdot v(t)\mathrm{d}F_Y(t) \\
&= \int \frac{w(t)}{\int w(t)\mathrm{d}F_Y(t)} \cdot [v^*(t) + v(\bar{y})]\mathrm{d}F_Y(t) \\
&= \int \frac{w(t)}{\int w(t)\mathrm{d}F_Y(t)} \cdot v^*(t)\mathrm{d}F_Y(t) + v(\bar{y})
\end{aligned}$$

As $0 < a \leqslant w \leqslant b$, $\frac{w(t)}{\int w(t)\mathrm{d}F_Y(t)} \leqslant \frac{b}{a} < \alpha$. And since $v^* \leqslant 0$, we get, similarly to the proof of claim 1, that

$$\begin{aligned}
V(Y) &\geqslant \alpha \int v^*(t)\mathrm{d}F_Y(t) + v(\bar{y}) \\
&= \alpha \int [v(t) - v(\bar{y})]\mathrm{d}F_Y(t) + v(\bar{y}) \\
&= \alpha\mathrm{E}[v(Y)] - \beta v(\bar{y}) \quad \blacksquare
\end{aligned}$$

Proof of Claim 3: To see that $\mathrm{E}[v(Y)] \geqslant V(Y)$, note that similarly to the proof of claim 2, in the DA model the utilities of all the outcomes that are strictly preferred to Y are multiplied by $\frac{1}{1+bF_Y(c(Y))} \leqslant 1$, while all other utilities are multiplied by $\frac{1+b}{1+bF_Y(c(Y))} \geqslant 1$ (note that $\int \gamma(t,b,c(Y))\mathrm{d}F_Y(t) = 1$). To show that there exist $\alpha \geqslant 1$ and $\beta \geqslant 0$ such that $V(Y) \geqslant \alpha \mathrm{E}[v(Y)] - \beta v(\bar{y})$, observe that $\gamma(t,b,c(Y)) \leqslant 1+b$ and proceed as in the RDU model with $\alpha = 1+b$ and $\beta = b$. ■

Proof of Claim 4: We consider wlg finite lotteries of the form $Y = (y_1, \frac{1}{n}; \ldots; y_n, \frac{1}{n})$ where $y_1 \leqslant \ldots \leqslant y_n$ and start with the inequality $V(Y) \leqslant \mathrm{E}[v(Y)]$. Since $v(y) = \psi(y,y)$, we get

$$\begin{aligned}
V(Y) &= \frac{1}{n^2}\left(\sum_i \psi(y_i,y_i) + \sum_i \sum_{j>i} [\psi(y_i,y_j)) + \psi(y_j,y_i)]\right) \\
&\leqslant \frac{1}{n^2}\left(\sum_i v(y_i) + \sum_i \sum_{j>i} [v(y_i) + v(y_j)]\right) \\
&= \frac{1}{n^2}\left(\sum_i v(y_i) + (n-1)\sum_i v(y_i)\right) = \mathrm{E}[v(Y)]
\end{aligned}$$

where the inequality follows by the condition $\psi(x,x) + \psi(y,y) \geqslant \psi(x,y) + \psi(y,x)$.

Next we show that $V(Y) \geqslant 2\mathrm{E}[v(Y)] - v(\bar{y})$. Recall that $\bar{y} = y_n$. Then

$$\begin{aligned}
V(Y) &= \frac{1}{n^2}\sum_i \sum_j \psi(y_i,y_j) \\
&= \frac{1}{n^2}\left[\sum_i \psi(y_i,y_i) + 2\sum_j \sum_{i>j} \psi(y_i,y_j)\right] \\
&\geqslant \frac{1}{n^2}\left[\sum_i v(y_i) + 2\sum_j \sum_{i>j} v(y_j)\right] \\
&= \frac{1}{n^2}\left[(2n-1)v(y_1) + (2n-3)v(y_2) + \ldots + v(y_n)\right]
\end{aligned}$$

$$\begin{aligned}
&\geqslant 2\mathrm{E}[v(Y)] - \frac{1}{n^2}\sum_{j=1}^{n}(2j-1)v(\bar{y}) \\
&= 2\mathrm{E}[v(Y)] - v(\bar{y})
\end{aligned}$$

The first inequality follows by the fact that for $i > j$, $y_i \geqslant y_j$ and by the monotonicity of ψ while the second inequality follows by the fact that for all i, $v(\bar{y}) \geqslant v(y_i)$. ■

Proof of Theorem 2: Since for all Y, $c(Y) = \min_{u\in\mathcal{W}} c_u(Y)$, it follows that for all $u \in \mathcal{W}$ (see eq. (1)),

$$\begin{aligned}
c(Q^n_{CI}) &= c\left(c(Y_{n1}), \mu_{n1}; \ldots; c(Y_{nH}), \mu_{nH}\right) \\
&\leqslant c\left(c_u(Y_{n1}), \mu_{n1}; \ldots; c_u(Y_{nH}), \mu_{nH}\right) \\
&\leqslant c_u\left(c_u(Y_{n1}), \mu_{n1}; \ldots; c_u(Y_{nH}), \mu_{nH}\right) \\
&= c_u(Q^n_{CI}) \\
&= c_u(Q^n_{R})
\end{aligned}$$

where the last equality follows from a property of the EU model. As $c(Q^n_R) = \min_{u\in\mathcal{W}} c_u(Q^n_R)$, we get $c(Q^n_{CI}) \leqslant c(Q^n_R)$ and hence

$$\lim_{n\to\infty}\frac{c(Q^n_{CI})}{n} \leqslant \lim_{n\to\infty}\frac{c(Q^n_R)}{n} \tag{13}$$

The set $\mathcal{W}$ is generated by a finite set of the utility functions $u_1, \ldots, u_\ell$. Let $r(x) = \max_j \left\{-\frac{u_j''(x)}{u_j'(x)}\right\}$. Define as in Pratt [23]

$$w(x) = \int_{-\infty}^{x} e^{-\int_{-\infty}^{z} r(t)dt} dz$$

By construction, $r(x) = -\frac{w''(x)}{w'(x)}$ and observe that $\lim_{x\to-\infty} r(x) = \bar{a}$. Let $c_w(Q^n_{CI}) = c_w(Q^n_R)$ denote the certainty equivalents of Q^n_{CI} and Q^n_R using the expected utility of w. Let $u = \sum \zeta_j u_j \in \mathcal{W}$ where $\sum \zeta_j = 1$ and $\zeta_1, \ldots, \zeta_\ell \geqslant 0$. Then

$$-\frac{u''(x)}{u'(x)} = -\frac{\sum \zeta_j u_j''(x)}{\sum \zeta_j u_j'(x)} = \frac{\sum \zeta_j u_j'(x)\left(-\frac{u_j''(x)}{u_j'(x)}\right)}{\sum \zeta_j u_j'(x)} \leqslant r(x)$$

It thus follows that w is more risk averse than all $u \in \mathcal{W}$, hence for all such u, $c_w(Y_j^n) \leqslant c_u(Y_j^n)$ and similarly to the argument above,

$$\begin{aligned} c_w(Q^n_{CI}) &= c_w\left(c_w(Y_{n1}), \mu_{n1}; \ldots; c_w(Y_{nH}), \mu_{nH}\right) \\ &\leqslant c_w\left(c_u(Y_{n1}), \mu_{n1}; \ldots; c_u(Y_{nH}), \mu_{nH}\right) \\ &\leqslant c_u\left(c_u(Y_{n1}), \mu_{n1}; \ldots; c_u(Y_{nH}), \mu_{nH}\right) = c_u(Q^n_{CI}) \end{aligned}$$

It thus follows that $c_w(Q^n_{CI}) \leqslant \min_{u \in \mathcal{W}} \left\{c_u(Q^n_{CI})\right\} = V(Q^n_{CI}) = c(Q^n_{CI})$ (recall that $V(\delta_z) = z$). Let c satisfy $\varphi_{\mathcal{W}}(c) = \mathrm{E}[\varphi_{\mathcal{W}}(Q_R)]$. By [26, Lemma 6], $\lim_{n\to\infty} \frac{c_w(Q^n_R)}{n} = c$ and hence, since $c_w(Q^n_R) = c_w(Q^n_{CI}) \leqslant c(Q^n_{CI})$,

$$c \leqslant \lim_{n\to\infty} \frac{c(Q^n_{CI})}{n} \tag{14}$$

Next, let $\bar{u}$ denote a utility in $\mathcal{W}$ satisfying $\lim_{x\to-\infty} -\frac{\bar{u}''(x)}{\bar{u}'(x)} = \bar{a}$. Again by [26, Lemma 6], $\lim_{n\to\infty} \frac{c_{\bar{u}}(Q^n_R)}{n} = c$. As $c(Q^n_R) = \min_{u\in\mathcal{W}} c_u(Q^n_R) \leqslant c_{\bar{u}}(Q^n_R)$, we get

$$\lim_{n\to\infty} \frac{c(Q^n_R)}{n} \leqslant c \tag{15}$$

Combining (13), (14) and (15)

$$c \leqslant \lim_{n\to\infty} \frac{c(Q^n_{CI})}{n} \leqslant \lim_{n\to\infty} \frac{c(Q^n_R)}{n} \leqslant c$$

and hence $\lim_{n\to\infty} \frac{c(Q^n_{CI})}{n} = \lim_{n\to\infty} \frac{c(Q^n_R)}{n} = c$.

The conclusion of the proof is the same as the conclusion of the proof of Theorem 1 with $\varphi_{\mathcal{W}}$ replacing φ_v. ■

References

[1] Abdellaoui, M., P. Klibanoff, and L. Placido, 2015. "Experiments on compound risk in relation to simple risk and to ambiguity," *Management Science* 61:1306–1322.

[2] Border, K.C. and U. Segal, 1994. "Dutch book arguments and subjective probability," *Economic Journal*, 104:71–75.

[3] Border, K.C. and U. Segal, 2002. "Coherent odds and subjective probability," *Journal of Mathematical Psychology*, 46:253–268.

[4] Cerreia-Vioglio, S., D. Dilleberger, and P. Ortoleva, 2015. "Cautious expected utility and the certainty effect," *Econometrica* 83:693–728.

[5] ———, 2020: "An explicit representation for disappointment aversion and other betweenness preferences," *Theoretical Economics*, forthcoming.

[6] Chateauneuf, A. and M. Cohen, 1994. "Risk seeking with diminishing marginal utility in a non-expected utility model," *Journal of Risk & Uncertainty* 9:77–91.

[7] Chew, S.H., 1983. "A generalization of the quasilinear mean with applications to the measurement of income inequality and decision theory resolving the Allais paradox," *Econometrica* 51:1065–1092.

[8] Chew, S.H., L.G. Epstein, and U. Segal, 1991. "Mixture symmetry and quadratic utility," *Econometrica* 59:139–163.

[9] Chew, S.H., E. Karni, and Z. Safra, 1987. "Risk aversion in the theory of expected utility with rank-dependent probabilities," *Journal of Economic Theory* 42:370–381.

[10] Chew S.H., B. Miao, and S. Zhong, 2017. "Partial Ambiguity," *Econometrica* 85:1239–1260.

[11] Epstein L.G. and Y. Halevy, 2021. "Hard-to-interpret signals," mimeo.

[12] de Finetti, B., 1964. "Foresight: Its logical laws, its subjective sources," In H. E. Kyburg, Jr., & H. E. Smokler (Eds.), *Studies in subjective probability* (pp. 57—118). New York: Wiley.

[13] De Finetti, B, 1987. "Probabilities of probabilities: A real problem or a misunderstanding?" in *New Developments in the Applications of Bayesian Methods*, ed. by A. Aykac and C. Brumat. Amsterdam: North Holland.

[14] Gillen, B., E. Snowberg, and L. Yariv, 2019: "Experimenting with measurement error: Techniques with applications to the Caltech Cohort Study," *Journal of political economy* 127:1826–1863.

[15] Gul, F., 1991. "A theory of disappointment aversion," *Econometrica* 59:667–686.

[16] Halevy, Y., 2007. "Ellsberg revisited: An experimental study," *Econometrica*, 75:503–536.

[17] Kahneman, D. and A. Tversky, 1979. "Prospect theory: An analysis of decision under risk," *Econometrica* 47:263–291.

[18] Kreps, D.M. and E.L. Porteus, 1978. "Temporal resolution of uncertainty and dynamic choice theory," *Econometrica* 46:185–200.

[19] Machina, M.J., 1982. " 'Expected utility' analysis without the independence axiom," *Econometrica* 50:277–323.

[20] Machina, M.J., 1989. Dynamic consistency and non-expected utility models of choice under uncertainty. *J. Econ. Literature* 27:1622–1668.

[21] McClennen, E.F., 1990. *Rationality and Dynamic Choice*. Cambridge: Cambridge University Press.

[22] Markowitz, H., 1959. *Portfolio selection: Efficient diversification of investments*. New Haven: Yale U. Press.

[23] Pratt, J.W., 1964. "Risk aversion in the small and in the large," *Econometrica* 32:122–136.

[24] Quiggin, J., 1982. "A theory of anticipated utility," *Journal of Economic Behavior and Organization* 3:323–343.

[25] Raiffa, H., 1968. *Decision analysis: Introductory lectures on choices under uncertainty*. Reading, MA: Addison Wesley.

[26] Safra, Z. and U. Segal, 2022. "A lot of ambiguity," *Journal of Economic Theory*, 200:105393.

[27] Samuelson, P.A., 1952. "Probability, Utility and the Independence Axiom," *Econometrica*, 20:670–678.

[28] Segal, U., 1990. "Two stage lotteries without the reduction axiom," *Econometrica* 58:349–377.

[29] Tversky, A. and D. Kahneman, 1992. "Advances in prospect theory: Cumulative representation of uncertainty," *Journal of Risk and Uncertainty* 5:297–323.