Nonemptiness of the f-Core Without Comprehensiveness*

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Abstract

Kaneko and Wooders (1986) showed under general conditions that an atomless NTU game with finite types of players has a core allocation when coalitions have a finite number of players. In this paper, we provide a direct proof of the above result using Kakutani's fixed point theorem when the sizes of coalitions are not only finite but also bounded above. This condition simplifies the presentation of the model and the existence proof. Most importantly, we can drop the comprehensiveness assumption, allowing for a much wider applicability of the result for matching problems, as well as for hedonic coalition formation problems. Additionally, without comprehensiveness, f-core allocations might not possess equal-treatment in payoffs for the same type. We also note that the nonemptiness of the core of NTU games by Scarf (1971) can be derived from our result as a corollary.

1 Introduction

In their influential paper, Kaneko and Wooders (1986) proved that the core of NTU games with a continuum of players of finite types is nonempty when the cardinality of admissible coalitions is finite under very general conditions. [1] With their *per capita boundedness* condition (see Wooders 1983, and Kaneko and Wooders 1986) or *small coalition effectiveness* condition (see Wooders 2008), their general proofs allow for unbounded size coalitions as long as they are finite. However, Kaneko and Wooders (1986) utilizes an ϵ-core existence result of a large replica game in Wooders (1983), and takes a limit to get their nonemptiness result in an atomless game: both papers' proofs are involved and quite lengthy.

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[1]One of the most recent developments in the finite core (the f-core) literature is found in Allouch and Wooders (2017).

In this paper, imposing a uniform upper bound in coalition sizes, we provide a direct proof of nonemptiness of the core using Kakutani's fixed point theorem so that their important theorem is easily accessible to more application-oriented researchers.

Moreover, this direct proof allows us to drop the comprehensiveness assumption entirely for nonemptiness of the f-core. This generalization broadens the applicability of our nonemptiness result of the f-core to a significantly wider class of problems such as matching problems in large markets: for example, couples or more generally, preferences over colleagues in one-to-many matching problems, hedonic games, and network formation problems when the size of each component's diameter is bounded above by a finite number.[2] Interestingly, in these cases, the equal treatment (in payoffs) property for players of the same type can be violated in every f-core allocation.

Our results are applicable to the models in the literature of matchings with atomless players such as Legros and Newman (1996), Konishi (2013), Gersbach, and Haller, and Konishi (2015), and Chade and Eeckhout (2020) as well as to atomless versions of the standard matching and hedonic problems such as Alkan (1986), Dutta and Masso (1997), Konishi, Quint, and Wako (2001), Banerjee, Konishi, and Sonmez (2001), and Bogomolnaia and Jackson (2002). We also discuss applications of our results to Scarf's (1971) nonemptiness result for the core of NTU games. We can also relate our results with the ones in Konishi, Pan, and Simeonov (2023) that analyze a team competition problem in a large market in the presence of moral hazard, showing the existence of a free-entry equilibrium of a team formation game.

The rest of the paper is organized as follows. Section 2 introduces a simplified version of the Kaneko-Wooders model and assumptions. Section 3 presents an atomless player version of a popular roommate example in a one-sided matching problem, and discusses how the f-core looks like in this example. Section 4 proves our main theorems. Theorem 1 shows that with the comprehensiveness condition, the equal-treatment f-core is nonempty. In contrast, our main result, Theorem 2, proves the nonemptiness of f-core without comprehensiveness, but there can be player types who are treated unequally in an f-core allocation.

2 The Model

There is a set of player types T, each of which has a continuum of atomless players of measure $\bar{\nu}_t > 0$ for each $t \in T$. Each coalition type γ is described by its membership profile, $(m_t^\gamma)_{t\in T}$, where $m_t^\gamma \in \mathbb{Z}_+$ is the number of type t players in coalition γ. Let the set of all admissible coalition types be Γ.

Let $T^\gamma = \{t \in T : m_t^\gamma > 0\}$. For each coalition type $\gamma \in \Gamma$, a feasible payoff allocation for types in T^γ is $u^\gamma = (u_t)_{t\in T^\gamma}$, and the collection of all feasible payoff vectors for coalition type γ is denoted by $V^\gamma \subset \mathbb{R}^{T^\gamma}$.[3]

[2] See Jackson (2008).

[3] For simplicity of presentation, we focus on equal treatment allocation of the same types in the same coalition. Regarding this property, see Example 2.

Our NTU model G is summarized by a list $G = (T, \Gamma, (\bar{\nu}_t)_{t \in T}, (V^\gamma)_{\gamma \in \Gamma})$. Let $\Gamma^t \equiv \{\gamma \in \Gamma : m^\gamma_{t'} = 0 \ \forall t' \neq t\}$, which is the set of coalition types that consists of only type t players. For $\gamma \in \Gamma^t$, let $\underline{u}^\gamma_t$ be the smallest upper bound such that $\underline{u}^\gamma_t \geq u_t$ for all $u_t \in V^\gamma$. For each $t \in T$, let $\underline{u}_t \equiv \max_{\gamma \in \Gamma^t} \underline{u}^\gamma_t$, this is the payoff guaranteed for type t player in a core allocation (individual rationality). Without loss of generality, for all $\gamma \in \Gamma$, we translate V^γ so that the **individually rational payoff for type** t is positive: i.e. $\underline{u}_t > 0$ for all $t \in T$.

(A1) T is a finite set

(A2) $V^\gamma - \mathbb{R}^{T^\gamma}_+ = V^\gamma$ for all $\gamma \in \Gamma$ (Comprehensiveness)

(A3) $V^\gamma \cap \mathbb{R}^{T^\gamma}_+$ is compact

(A4) Measure Consistency

(A5) There is $K \in \mathbb{Z}_{++}$ such that for all $\gamma \in \Gamma$, $0 < \sum_{t \in T} m^\gamma_t \leq K$ holds.

Assumptions (A1)-(A4) are employed in Kaneko and Wooders (1986). For the last technical condition (A4), see Kaneko and Wooders (1986)[4].

Our only simplification assumption of this paper is (A5): Kaneko and Wooders (1986) assume a weaker assumption, *per capita boundedness*. Note that (A1) and (A5) together imply that Γ is a finite set. We also introduce a slightly stronger version of (A2), which corresponds to strong monotonicity in money.

(A2') There is $b > 0$ such that for all $u^\gamma \in V^\gamma$, all $t \in T^\gamma$, all $c \in (0, u^\gamma_t)$, $(u^\gamma_t - c, (u^\gamma_{t'} + bc)_{t' \neq t}) \in V^\gamma$ (Strong Comprehensiveness).

This strengthened assumption (A2') plays a key role in achieving the equal-treatment property of the f-core: under (A2'), every f-core allocation treats the same type players equally in terms of their payoffs.

A **feasible assignment** for G is a list $((\nu^\gamma_t)_{t \in T, \gamma \in \Gamma})$ such that (a) $\frac{\nu^\gamma_t}{m^\gamma_t} = \frac{\nu^\gamma_{t'}}{m^\gamma_{t'}}$ for all $t, t' \in T$ and $\gamma \in \Gamma$, and (b) $\sum_{\gamma \in \Gamma} \nu^\gamma_t = \bar{\nu}_t$ for all $t \in T$. A **feasible allocation** for G is a list $((\nu^\gamma_t)_{t \in T, \gamma \in \Gamma}, (u^\gamma_t)_{t \in T^\gamma, \gamma \in \Gamma})$ such that (i) list $((\nu^\gamma_t)_{t \in T, \gamma \in \Gamma})$ is a feasible assignment, and (ii) $(u^\gamma_t)_{t \in T^\gamma} \in V^\gamma$ for all $\gamma \in \Gamma$. Let $T^\gamma \equiv \{t \in T : m^\gamma_t > 0\}$. An **f-core allocation** for G is a feasible allocation for G such that there is no pair $(\tilde{\gamma}, (\tilde{u}^{\tilde{\gamma}}_t)_{t \in T^{\tilde{\gamma}}})$ such that (i) $\left(\tilde{u}^{\tilde{\gamma}}_t\right)_{t \in T^{\tilde{\gamma}}} \in V^{\tilde{\gamma}}$, and (ii) for all $t \in T^{\tilde{\gamma}}$, $\tilde{u}^{\tilde{\gamma}}_t > u^\gamma_t$ holds for some γ with $\nu^\gamma_t > 0$. The **f-core** for G is a collection of all f-core allocations. An **equal-treatment f-core allocation** for G is a pair of a feasible assignment and payoff vector $((\nu^\gamma_t)_{t \in T, \gamma \in \Gamma}, (u^*_t)_{t \in T})$ such that (1) $(u^*_t)_{t \in T^\gamma} \in V^\gamma$ for all $\gamma \in \Gamma$ such that $\nu^\gamma_t > 0$ for all $t \in T^\gamma$, and (2) there is no $\gamma \in \Gamma$ and $(u'_t)_{t \in T^\gamma} \in V^\gamma$ such that $u'_t > u^*_t$ for all $t \in T^\gamma$.

[4]Condition (A4) is a technical but economically sensible condition (see the discussion in Kaneko and Wooders, 1986, pp. 108-109). It requires that measures of measurable sets are preserved in one to one mappings, which allows for intuitive handlings of matchings in an infinite world.

The **equal-treatment f-core** for G is a collection of all equal-treatment f-core allocations. Clearly, an equal-treatment f-core allocation for G is an f-core allocation for G as well.

The results of this paper are as follows:

Theorem 1. The equal-treatment f-core is nonempty under (A1), (A2), (A3), (A4), and (A5).

Theorem 2. The f-core is nonempty under (A1), (A3), (A4), and (A5).

Theorem 3. The f-core and the equal-treatment f-core are equivalent under (A1), (A2'), (A3), (A4), and (A5).

The differences between these theorems come from the assumptions around (A2), "Comprehensiveness." Although the main theorem is Theorem 2, the same type players might get different payoffs in every f-core allocation. In the following section, we present two simple educational examples, providing detailed analyses.

3 Examples

Here, we present two examples to illustrate our results before we present formal proofs. First, consider a continuum version of a roommate example in a hedonic game (Banerjee et al. 2001).

Example 1. Let $T = \{1, 2, 3\}$ and $K = 2$. There are only 6 feasible coalitions, and players' payoff vector in each coalition is determined uniquely (hedonic game): $(u_1, u_2) = (3, 2)$ for coalition $\{1, 2\}$; $(u_2, u_3) = (3, 2)$ for coalition $\{2, 3\}$; $(u_3, u_1) = (3, 2)$ for coalition $\{3, 1\}$; $u_t = 1$ for single-person coalition $\{t\}$ for all $t = 1, 2, 3$. Coalition $\{1, 2, 3\}$ is infeasible. If there are only three atomic players, it is clear that there is no core allocation. Consider an allocation (partition structure) $\{\{1, 2\}, \{3\}\}$. In this case, players get $(u_1, u_2, u_3) = (3, 2, 1)$. Clearly, players 2 and 3 deviate from it, generating $\{\{1\}, \{2, 3\}\}$ with $(u'_1, u'_2, u'_3) = (1, 3, 2)$. Since players are symmetric, there is no core allocation in this atomic model.

Here, we will consider an atomless model. Let $\bar{\nu}_t = 1$ for all $t = 1, 2, 3$. There are only one- or two-person coalitions. We first assume (A2) comprehensiveness (or *free disposal* of payoffs). Let's take comprehensive covers of the original payoffs. We have $\bar{V}^{\{1,2\}} = \big\{(u_1, u_2) \in \mathbb{R}^2 : u_1 \leq 3, u_2 \leq 2\big\}$, $\bar{V}^{\{2,3\}} = \big\{(u_2, u_3) \in \mathbb{R}^2 : u_2 \leq 3, u_3 \leq 2\big\}$, $\bar{V}^{\{3,1\}} = \big\{(u_3, u_1) \in \mathbb{R}^2 : u_3 \leq 3, u_1 \leq 2\big\}$, and $\bar{V}^{\{t\}} = \{u_t \in \mathbb{R} : u_t \leq \underline{u}_t\}$ for all $t = 1, 2, 3$. Denote the weak Pareto frontier of $\bar{V}^{\{t,t+1\}}$ by $\partial\bar{V}^{\{t,t+1\}}$ (see Figure 1). Here, we claim that there is a unique equal-treatment f-core allocation of the game with comprehensive-covers (Theorem 1): $\nu_1^{\{1,2\}} = \nu_2^{\{1,2\}} = \nu_2^{\{2,3\}} = \nu_3^{\{2,3\}} = \nu_3^{\{3,1\}} = \nu_1^{\{3,1\}} = \frac{1}{2}$ and $u_t^* = 2$

for all $t = 1,2,3$. There are coalitions $\{1,2\}$, $\{2,3\}$, and $\{3,1\}$ with measure $\frac{1}{2}$ each, and each coalition offers (weakly suboptimal) payoff $(2,2)$ for its members. Note that there is no strictly improving coalitional deviation. It is because coalition $\{1,2\}$ improves type 1 player's payoff from 2 to 3, while type 2 player's payoff is unchanged. In our setting, there is no means to transfer utility across players in the same coalition (unlike (A2')), and thus there is no possible deviations from weakly Pareto-inferior allocation. Symmetrically, there is no possibility for any coalitional deviation to improve all players in the coalition. This equal-treatment f-core allocation is shown in Figure 1.

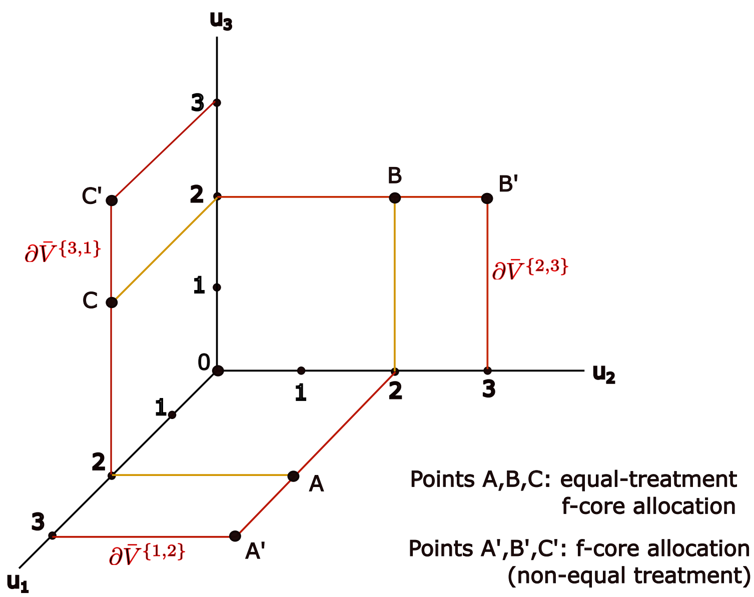

Figure 1: *The f-core allocations from Example 1 with and without comprehensiveness.*

Now, suppose that (A2) is dropped. Then, the above payoff vector is no longer feasible: $(u_1, u_2) = (2,2) \notin V^{\{1,2\}} = \{(3,2)\}$. Thus, there is no equal treatment f-core allocation. However, there is a weakly Pareto-improving payoff vector in the original hedonic game: $(u_1, u_2) = (3,2) \in V^{\{1,2\}}$. Since payoff vector $(u_1^*, u_2^*, u_3^*) = (2,2,2)$ cannot be blocked by any finite coalitions, $(u_1, u_2) = (3,2)$ cannot be blocked either. Thus, we have an f-core of the original hedonic game: $\nu_1^{\{1,2\}} = \nu_2^{\{1,2\}} = \nu_2^{\{2,3\}} = \nu_3^{\{2,3\}} = \nu_3^{\{3,1\}} = \nu_1^{\{3,1\}} = \frac{1}{2}$, $(u_1^{\{1,2\}}, u_2^{\{1,2\}}) = (3,2)$, $(u_2^{\{2,3\}}, u_3^{\{2,3\}}) = (3,2)$, and $(u_3^{\{3,1\}}, u_1^{\{3,1\}}) = (3,2)$. In this allocation, also shown in Figure 1, one half of each type players are getting less payoff than the other half. Despite of this apparent unequal treatment, there is no way for the worse-off players to form a strictly improving coalitional deviation. Consider players of type 1 in coalition type $\{3,1\}$. They are getting payoff 2, and they would rather get payoff 3 by forming a coalition $\{1,2\}$. However, it is not possible, since type 2 players who do not belong to coalition type $\{1,2\}$ are in coalition type $\{2,3\}$, getting $u_2^{\{2,3\}} = 3$. They are not attracted by a coalitional deviation offering them only payoff 2. This is how we obtain Theorem 2.

Our example has been assuming symmetry among the three types of players. What if the types of players do not have equal population? We finish our discussions on Example 1 by considering the following two situations:

1. (Small population difference) Let $\bar{\nu}_1 = 0.8$, $\bar{\nu}_2 = 1$, and $\bar{\nu}_3 = 1.2$. If the population differences are limited, then the f-core allocation requires that there is no single player. Suppose that a positive measure of type 3 are single. Then, if there is a positive measure of type 2 players who do not get the maximum payoff 3, then there will be coalitional deviations $\{2,3\}$. This implies that all type 2 players belong to coalition type $\{2,3\}$. This in turn implies that type 1 players can be matched with type 3 players, but there is not enough measure of type 3 players. However, if there is a positive measure of single type 1 players, type 3 players will dump type 2 players and deviate with these type 1 players. In order to avoid the existence of single players, the unique f-core allocation must be: $\nu_1^{\{1,2\}} = \nu_2^{\{1,2\}} = 0.3$, $\nu_2^{\{2,3\}} = \nu_3^{\{2,3\}} = 0.7$, and $\nu_3^{\{3,1\}} = \nu_1^{\{3,1\}} = 0.5$. Somewhat interestingly, type 2 has the largest fraction (70%) of players of getting higher payoff 3. Type 1 is the shortest side in terms of population, but the smallest fraction (37.5%) of players are getting the higher payoff in this case.

2. (Large population difference) Let $\bar{\nu}_1 = 1$, $\bar{\nu}_2 = 2$, and $\bar{\nu}_3 = 4$. The f-core allocation is: $\nu_2^{\{2,3\}} = \nu_3^{\{2,3\}} = 2$, $\nu_3^{\{3,1\}} = \nu_1^{\{3,1\}} = 1$, and $\nu_3^{\{3\}} = 1$. In this case type 2, who prefer to be paired with the most populous type 3 get to obtain their maximum payoff. This is on account that type 2 players would never want to match with type 1 given the overabundance of type 3's. Also note that in contrast to the example above, here type 3 players can remain single in an f-core allocation. Indeed, they get to participate in every type of coalition that is feasible for their type, which also results in a large variance of their payoffs. $\square$

Without (A2'), there may be an f-core allocation that does not satisfy equal-treatment even within the same coalition type. The following example illustrates this point.

Example 2. Consider a two-type game with $T = \{1,2\}$, $K = 2$, $\bar{\nu}^1 = 1$, $\bar{\nu}^2 = 1$, and three firm types: $\tilde{V}^{\{1\}} = \{u_1 \in \mathbb{R} : u_1 \leq 1\}$, $\tilde{V}^{\{2\}} = \{u_2 \in \mathbb{R} : u_2 \leq 1\}$, $\tilde{V}^{\{1,2\}} = \{(u_1,u_2) \in \mathbb{R}^2 : (u_1,u_2) \leq (3,2) \text{ or } (u_1,u_2) \leq (2,3)\}$. In this game, any allocation that assigns the two types of players to one of the two efficient points (i) $(u_1, u_2) = (3, 2)$ and (ii) $(u_1, u_2) = (2, 3)$ is an f-core allocation. In this case, (A2) is satisfied, but (A2') is violated. If (A2') is satisfied, a type 2 player in (i) can approach to a type 1 player in (ii) offering a payoff $u'_1 \in (2, 3)$. Then, by (A2'), this type 2 player can obtain a payoff higher than 2.□

4 Proofs of the Theorems

The main theorem of this paper is Theorem 2, but we utilize Theorem 1 in order to prove it. We will illustrate how the proof of Theorem 1 is constructed by using Example 1. Starting with the original hedonic game, we take comprehensive covers of the original payoff vectors: $\bar{V}^{\{1,2\}} = \{(u_1, u_2) \in \mathbb{R}^2 : u_1 \leq 3, u_2 \leq 2\}$, $\bar{V}^{\{2,3\}} = \{(u_2, u_3) \in \mathbb{R}^2 : u_2 \leq 3, u_3 \leq 2\}$, $\bar{V}^{\{3,1\}} = \{(u_3, u_1) \in \mathbb{R}^2 : u_3 \leq 3, u_1 \leq 2\}$, and $\bar{V}^{\{t\}} = \{u_t \in \mathbb{R} : u_t \leq 1\}$ for all $t = 1, 2, 3$. For each two person coalition, we now take the weak Pareto efficient set $\partial\bar{V}^{\{t,t+1\}} = \{(u_t, u_{t+1}) \in \mathbb{R}^2 : u_t \in [0, 3], u_{t+1} = 2\} \cup \{(u_t, u_{t+1}) \in \mathbb{R}^2 : u_t = 3, u_{t+1} \in [0, 2]\}$ for all $t = 1, 2, 3$. Let $\Gamma^t \equiv \{\{t, t-1\}, \{t, t+1\}, \{t\}\}$ be the set of coalitions that type t can join. In this example $\bar{\nu}^t = 1$ holds, and Δ^{Γ^t} denotes a population distribution simplex of type t. We construct a fixed point mapping that assigns (i) type t's total population to the highest payoff (possibly multiple) coalitions for each type t among all possible coalitions, and (ii) in each coalition $\{t, t+1\}$, if $\nu_t^{\{t,t+1\}} \neq \nu_{t+1}^{\{t,t+1\}}$ then the higher population type is assigned a zero payoff (if balanced, any payoff vector from $\partial\bar{V}^{\{t,t+1\}}$). Since each type t's individually rational payoff is $\underline{u}_t = 1 > 0$, if there is an over-populated type in a coalition, such an allocation cannot be a fixed point. Thus, in a fixed point, the population of each type is balanced in each coalition, and every type of player chooses the highest payoff coalition. Each unpopulated coalition type has a weak-Pareto efficient allocation that is not chosen by any type, which implies that such a coalition type cannot be used for a blocking coalition. Thus, the fixed point allocation achieves an equal-treatment f-core allocation. Theorem 1 is proved by the above argument with a formal treatment. (See Figures 2 and 3.)

Theorem 1. The equal-treatment f-core is nonempty under (A1), (A2), (A3), (A4), and (A5).

Proof of Theorem 1. First note that the individually rational payoff $\underline{u}_t$ is achievable in a coalition type $\gamma \in \Gamma^t$ by (A3). Let $\tilde{V}^\gamma \equiv V^\gamma \cap \mathbb{R}_+^{T^\gamma}$ for $\gamma \in \tilde{\Gamma} = \{\gamma \in \Gamma : V^\gamma \cap \mathbb{R}_+^{T^\gamma} \neq \emptyset\}$, and let $\tilde{V}^\gamma = \{0\}$ for all $\gamma \notin \tilde{\Gamma}$. We will replace Γ by $\tilde{\Gamma}$, since $\Gamma \backslash \tilde{\Gamma}$ is irrelevant. Let the weak Pareto frontier of $\tilde{V}^\gamma$ be $\partial\tilde{V}^\gamma \equiv \left\{u^\gamma \in \tilde{V}^\gamma : \{u^\gamma\} + \mathbb{R}_{++}^{T^\gamma} \cap \tilde{V}^\gamma = \emptyset\right\}$. For all $\gamma \in \tilde{\Gamma}$, let $\Delta^\gamma \equiv \{x^\gamma \in \mathbb{R}_+^{T^\gamma} : \sum_{t \in T^\gamma} x_t^\gamma = 1\}$, and consider a mapping $f^\gamma : \partial\tilde{V}^\gamma \to \Delta^\gamma$ such that $f^\gamma(u^\gamma) = \left(\frac{u_t}{\Sigma_{t' \in T^\gamma} u_{t'}}\right)_{t \in T^\gamma}$. This is a one-to-one continuous onto mapping in $V^\gamma \cap \mathbb{R}_+^{T^\gamma}$ due to comprehensiveness (A2).[5] For a visualization of this transfor-

[5]Strictly speaking, the inverse mapping of f^γ might be multi-valued at the border due to

mation please consult Figure 2 below. Let the inverse function of f^γ be u^γ : $\Delta^\gamma \to \partial\tilde{V}^\gamma$. This continuous mapping can be interpreted that $u^\gamma(x^\gamma)$ is a payoff vector for each abstract policy $x^\gamma \in \Delta^\gamma$. We will work on Δ^γ instead of $\partial\tilde{V}^\gamma$ to apply Kakutani's fixed point theorem.

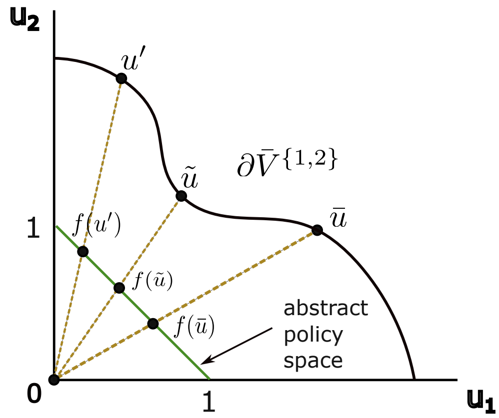

Figure 2: *The weak-Pareto frontier and its transformation onto abstract policy space (Theorem 1).*

For each $\gamma \in \tilde{\Gamma}$, let $\varphi^\gamma : \Pi_{t\in T}\left(\bar{\nu}_t \times \Delta^{\tilde{\Gamma}}\right) \twoheadrightarrow \Delta^\gamma$ be such that $\varphi^\gamma((\nu_t^\gamma)_{t\in T}) = \{(x_t^\gamma)_{t\in T} \in \Delta^\gamma : (1)\ x_t^\gamma = 0$ for all $t \notin T^\gamma$, (2) $x^\gamma = \arg\min_{x^\gamma} \sum_{t\in T^\gamma} x_t^\gamma \times \frac{\nu_t^\gamma}{m_t^\gamma}\}$. For each $t \in T$, let $\psi^t : \Pi_{\gamma\in\tilde{\Gamma}}(\Delta^\gamma) \twoheadrightarrow \bar{\nu}_t \times \Delta^{\tilde{\Gamma}}$ be such that $\psi^t((x^\gamma)_{\gamma\in\tilde{\Gamma}}) = \arg\max_{(\nu_t^\gamma)_{\gamma\in\tilde{\Gamma}}} \sum_{\gamma\in\tilde{\Gamma}} u_t^\gamma(x^\gamma)\nu_t^\gamma$. Let $\varphi : \Pi_{t\in T}\left(\bar{\nu}_t \times \Delta^{\tilde{\Gamma}}\right) \twoheadrightarrow \Pi_{\gamma\in\tilde{\Gamma}}(\Delta^\gamma)$ be a Cartesian product of φ^γs, and let $\psi : \Pi_{\gamma\in\tilde{\Gamma}}(\Delta^\gamma) \twoheadrightarrow \Pi_{t\in T}\left(\bar{\nu}_t \times \Delta^{\tilde{\Gamma}}\right)$ be a Cartesian product of ψ^ts. Letting ξ be a Cartesian product of φ and ψ, we have a fixed point mapping $\xi : \Pi_{\gamma\in\tilde{\Gamma}}(\Delta^\gamma) \times \Pi_{t\in T}\left(\bar{\nu}_t \times \Delta^{\tilde{\Gamma}}\right) \twoheadrightarrow \Pi_{\gamma\in\tilde{\Gamma}}(\Delta^\gamma) \times \Pi_{t\in T}\left(\bar{\nu}_t \times \Delta^{\tilde{\Gamma}}\right)$. Since sets $\Pi_{\gamma\in\tilde{\Gamma}}(\Delta^\gamma)$ and $\Pi_{t\in T}\left(\bar{\nu}_t \times \Delta^{\tilde{\Gamma}}\right)$ are nonempty, compact, and convex, and functions $\sum_{t\in T^\gamma} x_t^\gamma \times \frac{\nu_t^\gamma}{m_t^\gamma}$ and $\sum_{\gamma\in\tilde{\Gamma}} u_t^\gamma(x^\gamma)\nu_t^\gamma$ are continuous, and convex in u_t^γ and concave in ν_t^γ, respectively. Thus, ξ is nonempty-valued, upper hemi-continuous, and convex-valued. Since T and Γ are finite by (A1) and (A5), all the conditions of Kakutani's fixed point theorem is satisfied, and there is a fixed point $\left((\nu_t^\gamma)_{t\in T,\gamma\in\tilde{\Gamma}}, (x^\gamma)_{\gamma\in\tilde{\Gamma}}\right)$ of ξ.

Let $u_t^* \equiv \max_{\gamma\in\tilde{\Gamma}} u_t^\gamma(x^\gamma)$. Then, by construction of ψ^t, $\nu_t^\gamma > 0$ only if

'weak' Pareto frontier (there is at least one type who gets $u_t = 0$). However, we can truncate the abstract policy space slightly by a small $\epsilon > 0$ to ensure one-to-one and onto property of the mapping.

$u_t^{\gamma}(x^{\gamma}) = u_t^*$ for all $t \in T$. Thus, type t players get u_t^* almost everywhere. For all $\gamma \in \tilde{\Gamma}$, we have $\frac{\nu_{t'}^{\gamma}}{m_{t'}^{\gamma}} = \frac{\nu_t^{\gamma}}{m_t^{\gamma}}$ for all $t, t' \in T^{\gamma}$, since by the construction of φ^{γ}, if $\frac{\nu_{t'}^{\gamma}}{m_{t'}^{\gamma}} > \min_{t \in T^{\gamma}} \frac{\nu_t^{\gamma}}{m_t^{\gamma}}$ holds then $x_{t'}^{\gamma} = 0$ and $u_{t'}^{\gamma} = 0$ must occur, which cannot be a fixed point because for each $t \in T$, payoff $\underline{u}_t > 0$ is guaranteed by the individual rationality. Thus, $(\nu_t^{\gamma})_{t \in T, \gamma \in \tilde{\Gamma}}$ is a feasible assignment ($\nu_t^{\gamma} = 0$ for all $\gamma \notin \tilde{\Gamma}$ and all $t \in T$). Suppose that a pair $\left((\nu_t^{\gamma})_{t \in T, \gamma \in \tilde{\Gamma}}, (u_t^*)_{t \in T}\right)$ generated from the fixed point is not an f-core allocation. Then, there is a pair $\hat{\gamma} \in \tilde{\Gamma}$ and $(\tilde{x}^{\hat{\gamma}})_{t \in T^{\hat{\gamma}}} \in \Delta^{\hat{\gamma}}$ such that $u_t^{\hat{\gamma}}(\tilde{x}^{\hat{\gamma}}) > u_t^*$ for all $t \in T^{\hat{\gamma}}$. However, since $x^{\hat{\gamma}}$ is the fixed point allocation for $\hat{\gamma}$, $u_t^* \geq u_t^{\hat{\gamma}}(x^{\hat{\gamma}})$ holds for all $t \in T^{\hat{\gamma}}$. Since $u^{\hat{\gamma}}(x^{\hat{\gamma}}) \in \partial \tilde{V}^{\hat{\gamma}}$, there is no $\tilde{x}^{\hat{\gamma}}$ such that $u^{\hat{\gamma}}(\tilde{x}^{\hat{\gamma}}) \in \partial \tilde{V}^{\hat{\gamma}}$ with $u^{\hat{\gamma}}(\tilde{x}^{\hat{\gamma}}) \gg u^{\hat{\gamma}}(x^{\hat{\gamma}})$ by (A2). This is a contradiction. Hence, a pair $((\nu_t^{\gamma})_{t \in T, \gamma \in \Gamma}, (u_t^*)_{t \in T})$ generated from the fixed point $\left((\nu_t^{\gamma})_{t \in T, \gamma \in \tilde{\Gamma}}, (x_t^{\gamma})_{t \in T, \gamma \in \tilde{\Gamma}}\right)$ is an equal-treatment f-core allocation.□

Theorem 2 can be shown easily even if (A2) is violated by the argument in the latter half of Example 1.

Theorem 2. The f-core is nonempty under (A1), (A3), (A4), and (A5).

Proof of Theorem 2. Let $\bar{V}^{\gamma} \equiv \{u^{\gamma} \in \mathbb{R}^{T^{\gamma}} : u^{\gamma} \leq \tilde{u}^{\gamma} \text{ for some } \tilde{u}^{\gamma} \in V^{\gamma}\}$ be a comprehensive cover of V^{γ} for all $\gamma \in \tilde{\Gamma}$ (see Figure 3). Then, clearly (A2) satisfied for an NTU game $\bar{G} \equiv (T, \Gamma, (\bar{\nu}_t)_{t \in T}, (\bar{V}^{\gamma})_{\gamma \in \Gamma})$, and $\bar{G}$ has an equal treatment core allocation $((\nu_t^{\gamma})_{t \in T, \gamma \in \Gamma}, (u_t^*)_{t \in T})$ by Theorem 1. Let $\bar{u}^{\gamma} \in \partial \bar{V}^{\gamma}$ be a Pareto-efficient allocation within γ with $\bar{u}_t^{\gamma} \geq u_t^*$ for all $t \in T^{\gamma}$. Since $\bar{V}^{\gamma}$ is a comprehensive hull of V^{γ}, $\bar{u}^{\gamma} \in V^{\gamma}$ holds. Since $((\nu_t^{\gamma})_{t \in T, \gamma \in \Gamma}, (u_t^*)_{t \in T})$ is an equal-treatment f-core for $\bar{G}$, there is no pair $\hat{\gamma} \in \tilde{\Gamma}$ and $u^{\hat{\gamma}} \in \partial \bar{V}^{\hat{\gamma}}$ with $u_t^{\hat{\gamma}} > u_t^*$ for all $t \in T^{\hat{\gamma}}$, thus $((\nu_t^{\gamma})_{t \in T, \gamma \in \Gamma}, (\bar{u}_t^{\gamma})_{t \in T, \gamma \in \tilde{\Gamma}})$ has no pair $\hat{\gamma} \in \tilde{\Gamma}$ and $u^{\hat{\gamma}} \in \partial \bar{V}^{\hat{\gamma}}$ with $u_t^{\hat{\gamma}} > \bar{u}_t^{\hat{\gamma}}$ for all $t \in T^{\hat{\gamma}}$. Hence, $((\nu_t^{\gamma})_{t \in T, \gamma \in \Gamma}, (\bar{u}_t^{\gamma})_{t \in T, \gamma \in \tilde{\Gamma}})$ is feasible, thus is an f-core allocation.□

Comprehensiveness (A2) is easily achieved if an underlying economic problem has means of transferring payoffs across players. If there is a good (such as *money*) that every type of players has *strongly monotonic* preferences and is *essential* (any consumption bundle with zero money is the least preferred), Strict Comprehensiveness (A2') is satisfied, and we ∂V^{γ} becomes the Pareto-efficient frontier of V^{γ} instead of the weak-Pareto-efficient frontier.

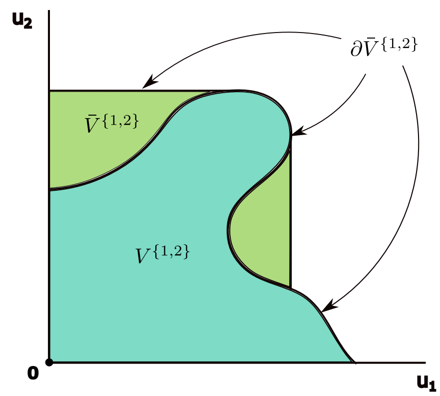

Figure 3: *The comprehensive hull and its weak-Pareto frontier (Theorem 2).*

Theorem 3. The f-core and the equal-treatment f-core are equivalent under (A1), (A2'), (A3), (A4), and (A5).

Proof of Theorem 3. First note that under (A2'), in any coalition $\gamma \in \Gamma$ with $\nu_t^\gamma > 0$ for $t \in T^\gamma$, two distinct allocations u^γ and $\tilde{u}^\gamma$ cannot coexist in an f-core allocation. Suppose not. Then there exist $\gamma \in \Gamma$ and $u^\gamma, \tilde{u}^\gamma \in V^\gamma$ such that $u^\gamma \neq \tilde{u}^\gamma$ and positive measures of players are assigned to these two allocations. Since $u^\gamma \neq \tilde{u}^\gamma$, there is $t \in T^\gamma$ with $u_t^\gamma < \tilde{u}_t^\gamma$ without loss of generality. Then, there exists $\hat{u}^\gamma \in V^\gamma$ such that $\hat{u}_{t'}^\gamma > \min\{u_t^\gamma, \tilde{u}_t^\gamma\}$ for all $t' \in T^\gamma$. This is a contradiction. Second, we will show that the following holds in any f-core allocation: In any two distinct coalitions $\gamma, \tilde{\gamma} \in \Gamma$ with $\nu_t^\gamma > 0$ and $\nu_t^{\tilde{\gamma}} > 0$ for $t \in T^\gamma$, and any $t \in T^\gamma \cap T^{\tilde{\gamma}}$, $u_t^\gamma = u_t^{\tilde{\gamma}}$. Suppose not, and assume that $u_t^\gamma < u_t^{\tilde{\gamma}}$ holds without loss of generality. Then, a coalition of coalition type $\tilde{\gamma}$ can kick out their type t member by replacing her with a type t player in a coalition of coalition type γ can improve all members of the new coalition by (A2'). Hence, there is no f-core allocation without equal-treatment property. We have completed the proof.□

5 Remarks

We list a few remarks on our assumptions and results here.

5.1 Strict *f*-Core and (A2')

Under (A2'), we can strengthen the equivalence theorem. Let a **strict f-core allocation** for G be a feasible allocation for G such that there is no pair $(\tilde{\gamma}, (\tilde{u}_t^{\tilde{\gamma}})_{t \in T^{\tilde{\gamma}}})$ such that (i) $\left(\tilde{u}_t^{\tilde{\gamma}}\right)_{t \in T^{\tilde{\gamma}}} \in V^{\tilde{\gamma}}$, and (ii) for all $t \in T^{\tilde{\gamma}}$, $\tilde{u}_t^{\tilde{\gamma}} \geq u_t^\gamma$

holds for some γ with $\nu_t^\gamma > 0$ with a strict inequality for at least one $t \in T^{\tilde{\gamma}}$. The **strict f-core** for G is a collection of all strict f-core allocations. Under (A2'), ∂V^γ becomes a Pareto-efficient frontier instead of a weak-Pareto-efficient frontier. Thus, as a corollary of Theorem 3, it is easy to see that we have the following result for the strict f-core.

Corollary 1. The strict f-core, the f-core, and the equal-treatment f-core are equivalent under (A1), (A2'), (A3), (A4), and (A5).

Without (A2'), the strict f-core may be empty. This can be shown by our Example 1.

Continuation of Example 1. The unique f-core allocation of the original game is $\nu_1^{\{1,2\}} = \nu_2^{\{1,2\}} = \nu_2^{\{2,3\}} = \nu_3^{\{2,3\}} = \nu_3^{\{3,1\}} = \nu_1^{\{3,1\}} = \frac{1}{2}$, $(u_1^{\{1,2\}}, u_2^{\{1,2\}}) = (3,2)$, $(u_2^{\{2,3\}}, u_3^{\{2,3\}}) = (3,2)$, and $(u_3^{\{3,1\}}, u_1^{\{3,1\}}) = (3,2)$. We claim that this is not a strict f-core allocation. Consider type 1 players in coalition type $\{3,1\}$. They can invite type 2 players who are already in coalition type $\{1,2\}$ simply switching partners. This way, these type 2 players are indifferent, but type 1 players in coalition type $\{3,1\}$ is better off. Thus, the allocation is not immune to a weakly-improving coalitional deviations.

5.2 Compact Type Sets

Kaneko and Wooders (1996) extended their nonemptiness of the f-core result in games with a compact set of types of atomless players. For this generalization, they strengthened their assumptions in two ways: One is (A2) Comprehensiveness to (A2') Strong Comprehensiveness which requires ∂V^γ has positive slopes uniformly bounded below everywhere (see Kaneko and Wooders 1996). The second one is to set a uniform upper bound for coalition size—exactly our (A5), instead of imposing milder conditions such as their small group effectiveness or per capita boundedness. Their proof is done by taking limits of the result by Kaneko and Wooders (1986) under more strict assumptions, thus our result can also be used to prove their theorem under (A2').

5.3 Scarf's (1971) Theorem

We can apply our Theorem 1 to prove Scarf's theorem (1971): the nonemptiness of the core of the standard NTU games under "Scarf-balancedness". Needless to say, there is only a single player for each type in a standard NTU game. A T-person NTU game is a list $(T, (V(S))_{S\subseteq T, S\neq\emptyset})$ such that $V(S) \equiv \left\{u \in \mathbb{R}^T : (u_t)_{t\in S} \in V^S,\ (u_t)_{t\in T\setminus S} \in \mathbb{R}^{T\setminus S}\right\}$ be a cylinder based on V^S for all $\emptyset \neq S \subseteq T$ (see, for example, Ichiishi, 1983). A **core allocation** of a T-person NTU game $(T, (V(S))_{S\subseteq T, S\neq\emptyset})$ is $(u_t^*)_{t\in T} \in V(T)$ such that there is no $S \in \Gamma = 2^T$ and $(u'_t)_{t\in T^S} \in V^{\overline{S}}$ such that $u'_t > u_t^*$ for all $t \in S$. We say that

$(T, (V(S))_{S \subseteq T, S \neq \emptyset})$ is **Scarf-balanced** if every balanced subfamily $\mathcal{B}$ of 2^T, it follows that $\cap_{S \in \mathcal{B}} V(S) \subseteq V(T)$. Scarf's theorem (1971) is as follows.

Corollary 2 (Scarf, 1971). Let $(T, (V(S))_{S \subseteq T, S \neq \emptyset}$ be an NTU game. The core of an NTU game $(T, (V(S))_{S \subseteq T, S \neq \emptyset})$ is nonempty if

(B1) $V(S) - \mathbb{R}^T_+ = V(S)$ for all $S \subseteq T, S \neq \emptyset$ (Comprehensiveness)

(B2) $V^S \cap \mathbb{R}^S_+$ is compact for all $S \subseteq T, S \neq \emptyset$

(B3) $(T, (V(S))_{S \subseteq T, S \neq \emptyset})$ is Scarf-balanced.

Proof. Consider the following special case of our problem in order to connect it with the standard NTU game: $\bar{\nu}_1 = ... = \bar{\nu}_T = 1$, $\Gamma \equiv \{S \subseteq T : S \neq \emptyset\}$, and $m^S_t = 1$ for all $t \in S \in 2^T \backslash \{\emptyset\} = \Gamma$ (where $T^S \equiv S$ for all $S \in \Gamma$). This special case is an atomless player version of a standard T-person NTU game $(T, (V(S))_{S \subseteq T, S \neq \emptyset})$. By Theorem 1, we have an f-core allocation $\left((\nu^S_t)_{t \in S, S \in \Gamma}, (u^*_t)_{t \in T}\right)$. Since $(\nu^S_t)_{t \in S, S \in \Gamma}$ is a feasible assignment, $\sum_{S \subseteq T, S \neq \emptyset} \nu^S_t = 1$ for all $t \in T$ and $\nu^S_t = \nu^S_r$ for all $t, r \in S$. Let $\nu^S = \nu^S_t$ for $t \in S \subseteq T$. This implies that $\mathcal{B} \equiv \left\{S \subseteq T : \nu^S > 0\right\}$ is a balanced family and $\left\{\nu^S : S \in \mathcal{B}\right\}$ is an associated balanced coefficients. Since $(u^*_t)_{t \in S} \in V(S)$ for any $S \in \mathcal{B}$, $(u^*_t)_{t \in T} \in \cap_{S \in \mathcal{B}} V(S)$ holds. By Scarf-balancedness, $(u^*_t)_{t \in T} \in V(T)$. By the definition of an f-core allocation, there is no $S \in \Gamma$ and $(u'_t)_{t \in T^S} \in V^S$ such that $u'_t > u^*_t$ for all $t \in T^S$. Hence, if an NTU game is Scarf-balanced, there is a core allocation $(u^*_t)_{t \in T} \in V(T)$.□

5.4 Competing Teams and Contracts

Alchian and Demsetz (1972) considered a team production problem in the presence of moral hazard in a partial equilibrium model, and Holmstrom (1982) showed that an efficient allocation is achievable depending on the class of contracts available for teams. We can illustrate how a team formation problem in a competitive environment can be incorporated in our framework to analyze an equilibrium team structure with optimal contracts, allowing for limited freedom for teams to choose their contracts. Let V^γ be a collection of all implementable payoff vectors for all available contracts for team-type γ. If the set V^γ is a compact set for all $\gamma \in \Gamma$, Theorem 2 shows that there is an f-core allocation.[6] That is, the f-core allocation is an allocation in which each team-type γ uses a contract, such that there is no feasible contract that can improve all members' payoffs. That is, an f-core allocation is an equilibrium competing contract structure—a list of team contracts that cannot be shaken by any other contracts by entrants with new contracts (Konishi, Pan, and Simeonov 2023). In addition, we can allow for wide-spread externalities due to market price changes—if

[6] Moral hazard problems may not necessarily have binding individual rationality constraints due to limited liability by the agent, and the comprehensiveness assumption could be violated. In such a case, an f-core allocation may not satisfy the equal-treatment property

market price changes, the set of achievable payoffs V^γ can change as well. Hammond, Kaneko, and Wooders (1989) and Kaneko and Wooders (1989) introduced widespread externalities to Kaneko and Wooders model (1986), and showed that the f-core is nonempty using the property that atomless coalitions' deviations do not affect the whole economy (in contrast, the Aumann core can be empty under widespread externalities due to the atomic impact of a large (positive-measure) coalition's deviation). Our fixed-point-based proof strategy turns out to be useful even under widespread externalities as is shown in Konishi, Pan, and Simeonov (2023).

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