Stability in Matching with Externalities: Pairs Competition and Oligopolistic Joint Ventures*

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Abstract

This paper presents one-to-one matching and assignment problems with externalities across pairs such as pairs figure skating competition and joint ventures in oligopolistic markets. In these models, players care not only about their partners but also which and how many rival pairs are formed. Thus, it is important for a deviating pair to know which matching will realize after it deviates from a matching (an effectiveness function) in order to define pairwise stable matching. Using a natural effectiveness function for such environments, we show that the assortative matching is pairwise stable. We discuss two generalizations of our model including intrinsic preferences on partners and pair-specific match qualities to see how our stability concept performs in these generalized models.

Keywords: one-to-one matching, matching with externalities, pairwise stable matching, coalition formation, group contest, joint ventures, myopia, farsightedness.

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1 Introduction

There is a huge literature on two-sided matching problems in both non transferable and transferable utility setups. Gale and Shapley (1962) introduced their celebrated marriage and college admission problems under non transferable utilities (NTU), and showed the nonemptiness of the set of (pairwise) stable matchings. In their assignment game, which is a transferable utility (TU) version of the marriage problem, Shapley and Shubik (1972) showed that the core is nonempty, and analyzed the structure of the core. In these problems, the set of pairwise stable matchings and the core are equivalent to each other, and thus stable matching and stable assignment became the central solution concepts.

Although the structure of stable matchings has been analyzed extensively following these two papers, one simplification assumption has been maintained: players only care about their partners, and the rest of the matching does not matter. However, it is easy to imagine matching problems in which the rest of the matching matters for every pair. For example, consider formation of pairs in sports competition, such as a pairs figure skating competition. Each participant may have preferences over her partner, but will also care about other pairs that are formed, since those other pairs represent the competition. For transferable utility problems, consider formation of a two-member joint ventures (say, between a marketing company and a technology company) in an industry. Although a joint venture's profit will be split between the constituent marketing and technology companies, the other ventures formed also matter as they also affect profits. Somewhat surprisingly, these problems with externalities have gone relatively unexamined. Hence, it is meaningful to introduce tractable models of matching with externalities and to provide reasonable solution concepts that are natural extensions of those used for matching problems without externalities.

Sasaki and Toda (1996) first introduced a one-to-one matching problem with externalities. Using a general framework that allows for any kind of externalities, they considered a set of admissible matchings which can be realized after a deviating pair is formed, and define stable matching by assuming that players expect the worst possible outcome among the admissible set. They showed that the admissible set needs to be the set of all matchings to assure the existence of stable matching, and proved that there always exists a Pareto-efficient stable matching. Hafalir (2007) imposed certain additional rationality constraints on players' expectations of which set of matchings might be realized by forming a pair, and showed the existence of stable matching under pessimism as in Sasaki and Toda (1996). Chen (2019) considered a specific example of Cournot oligopoly game played by joint ventures, and assumed that every potential partner for a company induces a unique consistent expectation for the realized matching. With this list of expectations for each possible pair, stable matching is defined as the outcome of this game. Chen identified conditions under which positively and negatively assortative matchings are stable.

In these papers, each player has expectations on the realization of a matching when she is partnered with each of the players on the other side of the market, and stable matching is built on these expectations. In contrast, this paper mirrors the original definition of pairwise stable matching in matching problems without externalities. Our pairwise stability starts with a matching and checks whether or not there is a pair of players with a profitable deviation away from the original matching. There is, however, a subtle issue in the presence of externalities—the rest of matching matters. Thus, we need to formulate the matching induced by the deviation of a pair from the original matching. We specify this using an effectiveness function, and consider two effectiveness functions specifically. The first is that after a deviation by a pair, the dumped partners stay single and no other player changes their partners. The second is that after a deviation by a pair, the dumped partners form a pair and all other players remain with their partners. The former effectiveness function is in the literature of theory of coalition formation, and is adopted to analyze convergence of a sequence of myopic deviations in a marriage problem by Roth and Vande Vate (1990).[1] The latter effectiveness function was proposed by Knuth (1976) in the context of the marriage problem in which all players are acceptable to all other players.

To see the difference between these two effectiveness functions, consider the example of a pairs figure skating competition. Suppose that there are three male and three female skaters, one with high, medium, and low ability in each gender. Moreover, suppose that there are complementarities in partners' abilities. Then, it is natural for them to have a (positively) assortative matching, since a high ability partner is always desirable. However, this assortative matching may not be pairwise stable in the Roth-Vande Vate sense. Consider a deviation by the high ability male and the

[1] Diamantoudi, Miyagawa, and Xue (2004) and Kojima and Unver (2008) also use the Roth-Vande Vate effectiveness function in the context of matching theory. This effectiveness function is a special case of the most standard transition functions in the theory of coalition formation (see Hart and Kurz, 1983, Bloch 1997, Yi 1997, Ray 2008, and Ray and Vohra 2014, among others).

medium ability female skaters. As a result, the high ability female and the medium ability male become single, leaving only two pairs left in the competition, improving the probability of winning for the deviating pair. In the assortative matching, the high and medium ability pairs need to compete hard, and the deviating pair can benefit by blocking the assortative matching. Thus, there may not be pairwise stable matching under the Roth-Vande Vate effectiveness function when pairs are better off by having a smaller number of competing pairs. In contrast, the assortative matching is pairwise stable under the Knuth effectiveness function, since the above deviation does not decrease the number of competing pairs, and thus does not benefit the high ability deviator. We call pairwise stability under the Knuth effectiveness function pairwise stability via swapping, and employ this solution concept to analyze our matching/assignment problems with externalities.

In this paper, we propose two natural one-to-one matching models in the presence of externalities with and without transfers between paired players: one is a pairs competition model we described above without transfers, and the other is an oligopolistic joint ventures between, say, a marketing company and a technology company with transfers (profits are shared by the companies). In both models, players care about not only who they are matched with but also the competition of competing pairs. We assume that players are vertically differentiated in their abilities. We analyze the properties of pairwise stable matchings and assignments via swapping, and show that in these two models, pairwise stability is supported by the assortative matching.

In a companion paper, Imamura and Konishi (2021) analyzed the pairs competition problem by using Harsanyi's (1974) indirect dominance with farsighted players. They use the largest consistent set (LCS) by Chwe (1994) as the solution concept, and show that LCS of the pairs competition problem can be a large set under the Roth-Vande Vate effectiveness function, while it is a singleton set of the assortative matching under the Knuth one. That is, under the Knuth effectiveness function, myopic pairwise stability and farsighted LCS are equivalent, uniquely pointing at the assortative matching.

The rest of the paper is organized as follows. We first provide a brief literature review. In Section 2, we introduce a one-to-one matching model with externalities. In Section 3, we introduce a pairs competition problem, in which after players are matched, the members of each pair choose effort noncooperatively to win in a Tullock contest. Players differ vertically in their abilities, and we show that a (positively) assortative matching is the unique pairwise stable matching via swapping. In Section 4, we consider an oligopolistic joint ventures problem, which is an assignment game version of the pairs competition model without endogenous efforts. We show that pairwise stable assignments can be supported only by the assortative matching, and characterize the one-side optimal stable assignment. In Section 5, we introduce personalized intrinsic utility from partners and heterogeneous match qualities of players, and show that pairwise stable assignment via swapping may not exist. Section 6 concludes.

1.1 A Brief Literature Review

There are three branches of literature that are related to the current paper. The first branch is the one of matching with externalities. Recently, a number of papers have been written in this field in addition to Sasaki and Toda (1996), Hafalir (2008), and Chen (2019). Mumcu and Saglam (2010), Fisher and Hafalir (2016), and Chade and Eeckout (2020) all dealt with one-to-one matching problems with externalities in different ways. Mumcu and Saglam (2010) introduced outside options, and Fisher and Hafalir (2016) and Chade and Eeckhout (2020) removed the impacts of pairwise deviations through externalities by imposing a behavioral assumption and by considering a continuum of atomless agents, respectively. Bando (2012, 2014), and Pycia and Yenmez (2015) considered one-to-many and many-to-many matchings, and analyzed the standard stability concept and its existence by imposing assumptions on agents' preferences.

Second, our paper belongs to the literature of the assortative matching. Becker (1973) introduced an assortative model of marriages. Banerjee, Konishi, and Sonmez (2001) extended Becker's assortative matching problem to hedonic coalition formation problems without externalities by defining a top coalition property, and proved the nonemptiness and uniqueness of the core.[2] Diamantoudi and Xue (2003) proved that under the top-coalition property, the largest consistent set coincides with a singleton core under the standard effectiveness function in the literature of coalition formation. Mauleon, Vannetelbosch, and Vergote (2011) derived the same result in the context of one-to-one matching under the Roth Vande Vate effectiveness function. Although our model has the same assortative structure, the results in the current paper and Imamura and Konishi (2021) differ substantially from the literature due to the externality.

Third, our pairs competition model is a special case of the literature of group contests. Nitzan

[2]See Bogomolnaia and Jackson (2002) and Leo et al. (2021) as well.

(1991) and Esteban and Ray (2001) established contest models played by groups in which players choose their efforts noncooperatively, making players' efforts subject to free-riding incentives. Kolmar and Rommeswinkel (2013) introduce the use of CES effort aggregation functions for teams' production function to capture effort complementarity. Konishi and Pan (2020, 2021) consider a group formation game of such a group contest assuming homogeneous players.[3] Kobayashi, Konishi, and Ueda (2021) and Konishi, Pan, and Simeonov (2021) introduce heterogenous players and heterogenous award-distribution rules to the CES group contest. Our pairs competition model belongs to this line of research, providing a convenient and tractable payoff structure in this matching problem with externalities.

2 One-to-One Matching Models with Externalities

Consider a contest played by pairs of male and female players. Let $M = \{m_1, ..., m_n\}$ and $W = \{w_1, ..., w_n\}$ be the sets of male and female players with $|M| = |W| = n$. Let $\mu : M \cup W \to M \cup W$ be a one-to-one matching of players: $\mu(\mu(x)) = x$ for all $x \in M \cup W$ such that if $\mu(m) \neq m$ then $\mu(m) \in W$, and if $\mu(w) \neq w$ then $\mu(w) \in M$. The set of all possible matchings is denoted by $\mathcal{M}$. A matching μ is **fully matched** if $\mu(x) \neq x$ for all $x \in M \cup W$. The set of all fully matched matchings is denoted by $\mathcal{M}^F$. Each player $x \in M \cup W$ has preference over matchings, which is denoted by a binary relation R_x on $\mathcal{M}$: player x weakly prefers μ' to μ if and only if $\mu' R_x \mu$ for any distinct matchings $\mu, \mu' \in \mathcal{M}$. Strict preference relation P_x is defined by: $\mu' P_x \mu$ if and only if $\mu' R_x \mu$ and $\neg \mu R_x \mu'$. We assume that *having no partner is the worst outcome*: for any $\mu, \mu' \in \mathcal{M}$ and $x \in M \cup W$, $\mu(x) = x$ and $\mu'(x) \neq x$ imply $\mu' P_x \mu$.

We add more structure to the model by assuming that agents m_i and w_i are endowed with *abilities* a_{m_i} or a_{w_i} for all $i = 1, ..., n$. We assume strict ordering over abilities: $a_{m_1} > a_{m_2} > ... > a_{m_n}$ and $a_{w_1} > a_{w_2} > ... > a_{w_n}$.

We will consider two effectiveness functions, $\to_S$ and $\rightrightarrows_S$ (Rosenthal 1972).[4] A matching μ' **is induced from** μ **by a pairwise deviation** $(m, w) \in M \times W$ if and only if (i) $\mu(m) \neq w$ and $\mu'(m) = w$; (ii) if m is paired under μ ($\mu(m) \neq m$) then m's partner $\mu(m)$ is single under μ'

[3] Bloch, Sanchez-Pages, and Soubeyran (2006) and Sanchez-Pages (2007a,b) are the first papers that considered group formation in contests.

[4] More specifically, these are single-valued effectiveness relations in Rosenthal (1972).

$(\mu'(\mu(m)) = \mu(m))$, and if w is paired under μ $(\mu(w) \neq w)$ then w's partner $\mu(w)$ is single under μ' $(\mu'(\mu(w)) = \mu(w))$; and (iii) for all $x \in M \cup W \setminus \{m, w, \mu(m), \mu(w)\}$, $\mu(x) = \mu'(x)$. This is denoted $\mu \to_{(m,w)} \mu'$. For completeness, we also define a player's deviation by terminating a pair, although being single is the worst outcome for any player. A matching μ' **is induced from** μ **by a single player deviation** $x \in M \cup W$ if and only if (i) $\mu'(\mu(x)) = \mu(x)$; and (ii) for all $y \in M \cup W \setminus \{x, \mu(x)\}$, $\mu(y) = \mu'(y)$. This is denoted $\mu \to_x \mu'$.

Since we assume that having no partner is always the worst possible outcome independent of how other players are matched, it may not make sense to expect a matching μ to include any singles as a result of a (pairwise) deviation. Thus, we assume that these two abandoned players form a pair—this is the most myopic response from the leftover players who do not want to be left alone (Knuth 1976). Assuming this reaction by the leftover players, a pairwise blocking generates swapping of their partners among the relevant four players. Suppose that $\mu \in \mathcal{M}^F$. A matching μ' **is induced from** μ **by a pairwise deviation** $(m, w) \in M \times W$ **via swapping** iff (i) $\mu(m) \neq w$; (ii) $\mu'(m) = w$ and $\mu'(\mu(m)) = \mu(w)$; and (iii) for all $x \in M \cup W \setminus \{m, w, \mu(m), \mu(w)\}$, $\mu(x) = \mu'(x)$. This effectiveness function is denoted $\mu \rightrightarrows_{(m,w)} \mu'$. Since a single player deviation from $\mu \in \mathcal{M}^F$ reverts to μ by matching up two singles induced by the deviation, we do not need to consider this case.

If there are no externalities the choice of effectiveness function does not make a difference. However, with externalities, their impacts are quite pronounced, which is shown in the next section.

3 Stable Matching in Pairs Competition

Imagine a figure skating pairs competition. We formulate a pairs competition as a team contest. Given a matching $\mu \in \mathcal{M}$, the pairs compete with each other as a team for a single prize: the players of the winning pair get payoff $V > 0$ each. Unmatched players cannot participate in the contest, obtaining a zero payoff. Thus, there are n potential teams, and for simplicity, we denote each team by its male player's number $i = 1, ..., n$. In this contest, each player x of a pair chooses his/her effort level e_x simultaneously and non-cooperatively. The ability parameters a_{m_i} and a_{w_i} describe the efficiency of their efforts in the pairs competition. If m_i is matched under μ

$(\mu(m_i) \in W)$, team i's members' efforts are aggregated by a CES function

$$Y_i = (a_{m_i}^{\sigma} e_{m_i}^{\sigma} + a_{\mu(m_i)}^{\sigma} e_{\mu(m_i)}^{\sigma})^{\frac{1}{\sigma}}, \tag{1}$$

where $0 < \sigma \leq 1$.[5] This CES aggregator function becomes a linear function (perfect substitutes) when $\sigma = 1$, and becomes a Cobb-Douglas function when $\sigma = 0$ in the limit. If m_i is unmatched $(\mu(m_i) = m_i)$, then $Y_i = 0$. Teams' aggregate effort vector $(Y_1, ..., Y_n)$ determines each team's winning probability. The winning probabilities of teams are determined by a Tullock-style contest: team i's winning probability is given by

$$\pi_i = \frac{Y_i}{\sum_{k=1}^{n} Y_k}. \tag{2}$$

The effort cost function is common and linear for every player x: $c_x(e_x) = e_x$. Therefore, the expected payoffs of player x in team i is

$$U_x = \pi_i V - e_x.$$

Each member in a team decides his/her effort level to maximize his/her expected payoff independently and simultaneously. Thus, there are free-riding incentives in a team. We assume that team i members regard the other groups' aggregate effort $Y_{-i} = \sum_{j \neq i} Y_j$ as given, and consider a Nash equilibrium of team i's effort contribution game as the best response of team i to the other teams' aggregate effort Y_{-i}.

For the time being, let's assume that all teams and all players make positive efforts. If so, the first-order condition of player x in team i $(x \in \{m_i, \mu(m_i)\})$ is

$$\frac{\partial U_x}{\partial e_x} = \frac{(a_x^{\sigma} e_x^{\sigma} + a_{\mu(x)}^{\sigma} e_{\mu(x)}^{\sigma})^{\frac{1}{\sigma}-1} a_x^{\sigma} e_x^{\sigma-1} Y_{-i}}{\left(\left(a_x^{\sigma} e_x^{\sigma} + a_{\mu(x)}^{\sigma} e_{\mu(x)}^{\sigma}\right)^{\frac{1}{\sigma}} + Y_{-i}\right)^2} V - 1 = 0.$$

By using $Y_i^{1-\sigma} = \left(a_x^{\sigma} e_x^{\sigma} + a_{\mu(x)}^{\sigma} e_{\mu(x)}^{\sigma}\right)^{\frac{1}{\sigma}-1}$, this can be rewritten as

$$(1 - \pi_i) \frac{1}{Y} Y_i^{1-\sigma} a_x^{\sigma} e_x^{\sigma-1} V - 1 = 0,$$

since $\frac{Y_{-i}}{Y} = 1 - \pi_i$. From this expression, we have

$$e_x^{1-\sigma} = Y_i^{1-\sigma} \left[(1 - \pi_i) \frac{1}{Y}\right] a_x^{\sigma} V$$

[5] Kolmar and Rommeswinkel (2013) call this CES function a group impact function.

and player x's equilibrium effort given Y_i and Y can be written as

$$e_x = Y_i \left[(1-\pi_i)\frac{1}{Y} \right]^{\frac{1}{1-\sigma}} a_x^{\frac{\sigma}{1-\sigma}} V^{\frac{1}{1-\sigma}}. \tag{3}$$

Raising this to the power of σ and then multiply it by a_x^σ,

$$a_x^\sigma e_x^\sigma = Y_i^\sigma \left[(1-\pi_i)\frac{1}{Y} \right]^{\frac{\sigma}{1-\sigma}} a_x^{\frac{\sigma}{1-\sigma}} V^{\frac{\sigma}{1-\sigma}}$$

is obtained (the power of a_x is calculated by $\frac{\sigma^2}{1-\sigma} + \sigma = \frac{\sigma}{1-\sigma}$). Substituting this back to (1), we obtain

$$Y_i = Y_i \left[(1-\pi_i)\frac{1}{Y} \right]^{\frac{1}{1-\sigma}} \left(a_x^{\frac{\sigma}{1-\sigma}} + a_{\mu(x)}^{\frac{\sigma}{1-\sigma}} \right)^{\frac{1}{\sigma}} V^{\frac{1}{1-\sigma}}$$

or

$$\frac{1}{A_i(\mu)} = \frac{Y_{-i}}{Y^2} V, \tag{4}$$

where $A_i(\mu) = \left(a_{m_i}^{\frac{\sigma}{1-\sigma}} + a_{\mu(m_i)}^{\frac{\sigma}{1-\sigma}} \right)^{\frac{1-\sigma}{\sigma}}$ stands for the *productivity of team i*. Let $A_j(\mu) = 0$ when $\mu(m_j) = m_j$. Summing the above up over all active teams, we have

$$\sum_{j=1}^{n} \frac{1}{A_j(\mu)} = \frac{n(\mu)-1}{Y} V$$

or

$$Y = \frac{n(\mu)-1}{\sum_{j=1}^{n} \frac{1}{A_j(\mu)}} V,$$

where $n(\mu)$ is the number of pairs under matching μ.

Let μ^* be a (positively) assortative matching: i.e., $\mu^*(m_i) = w_i$ for all $i = 1, ..., n$. We will impose the following assumption, which assures that all pairs will be active in making effort under any $\mu \in \mathcal{M}$.

Regularity Condition 1. *Under the assortative matching μ^*, the following inequality holds:*

$$\frac{n-1}{A_n(\mu^*)} \leq \sum_{i=1}^{n} \frac{1}{A_i(\mu^*)}.$$

Under the Regularity Condition 1, we have the following proposition.[6]

[6] This regularity condition is imposed for simplicity of the analysis. Konishi, Pan, and Simeonov (2021) considers a general team-size contest with flexible prize sharing rules, and allows for inactive teams (zero-effort and zero-winning probability teams by using a share-function approach by Cornes and Hartley, 2005).

Proposition 1. *For any $\mu \in \mathcal{M}$, there exists a unique equilibrium in the pairs competition model under the regularity condition 1. Team i's winning probability is*

$$\pi_i(\mu) = 1 - \frac{(n(\mu)-1)\frac{1}{A_i(\mu)}}{\sum_{j=1}^{n}\frac{1}{A_j(\mu)}},$$

agent $x \in \{m_i, \mu(m_i)\}$ of team $i = 1, ..., n$ obtains payoff

$$U_x = \begin{cases} \left[1 - \frac{(n(\mu)-1)\frac{1}{A_i(\mu)}}{\sum_{j=1}^{n}\frac{1}{A_j(\mu)}}\right]\left[1 - \frac{(n(\mu)-1)\frac{1}{A_i(\mu)}}{\sum_{j=1}^{n}\frac{1}{A_j(\mu)}}\left(\frac{a_x}{A_i(\mu)}\right)^{\frac{\sigma}{1-\sigma}}\right]V & \text{if } x \neq \mu(x) \\ 0 & \text{if } x = \mu(x) \end{cases},$$

by exerting effort

$$e_x = \begin{cases} \left[1 - \frac{(n(\mu)-1)\frac{1}{A_i(\mu)}}{\sum_{j=1}^{n}\frac{1}{A_j(\mu)}}\right]\left[\frac{(n(\mu)-1)\frac{1}{A_i(\mu)}}{\sum_{j=1}^{n}\frac{1}{A_j(\mu)}}\right]\left(\frac{a_x}{A_i(\mu)}\right)^{\frac{\sigma}{1-\sigma}}V & \text{if } x \neq \mu(x) \\ 0 & \text{if } x = \mu(x) \end{cases}.$$

Moreover, the equilibrium total efforts are

$$Y = \frac{n(\mu)-1}{\sum_{j=1}^{n}\frac{1}{A_j(\mu)}}V,$$

and

$$(n(\mu)-1)\frac{1}{A_i(\mu)} < \sum_{j=1}^{n}\frac{1}{A_j(\mu)}$$

holds for all $i = 1, ..., n$.

Remark 1. *We can interpret the formula for U_x in the following way:*

$$U_x = \underbrace{\left[1 - \frac{(n(\mu)-1)\frac{1}{A_i(\mu)}}{\sum_{j=1}^{n}\frac{1}{A_j(\mu)}}\right]}_{\text{winning probability}}\underbrace{\left[1 - \frac{(n(\mu)-1)\frac{1}{A_i(\mu)}}{\sum_{j=1}^{n}\frac{1}{A_j(\mu)}}\left(\frac{a_x}{A_i(\mu)}\right)^{\frac{\sigma}{1-\sigma}}\right]}_{\text{net benefits by taking effort disutility into account}}V$$

Note that $\frac{a_x}{A_i(\mu)}$ denotes player x's contribution to pair i by his/her ability, and the contents of the second bracket indicates that a higher ability player needs to suffer from a higher disutlity by exerting more effort than his/her lower ability partner. This can be regarded as a free-riding problem of the pairs competition problem.

We first consider a pairwise stability concept using the Roth-Vande Vate effectiveness function $\rightarrow_S$, which is commonly used in the literature of coalition formation. A matching μ is **pairwise**

stable if and only if (i) $\mu R_m \mu'$ or $\mu R_w \mu'$ for any pairwise deviations $(m, w) \in M \times W$ with $\mu \rightarrow_{(m,w)} \mu'$, and (ii) $\mu R_x \mu'$ for any single player deviation $x \in M \cup W$ with $\mu \rightarrow_x \mu'$. The following example shows that there may not be a pairwise stable matching.

Example 1. Consider a figure skating contest with $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Let $\sigma = \frac{1}{2}$, $a_{m_1} = a_{w_1} = 1$, $a_{m_2} = a_{w_2} = 0.9$, and $a_{m_3} = a_{w_3} = 0.7$. We calculate m_1's payoffs under the assortative matching and the one after he deviates with w_2:

(i) $\mu^* = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$:

$$U_{m_1}(\mu^*) = \left(1 - \frac{2 \times \frac{1}{2}}{\frac{1}{2} + \frac{1}{1.8} + \frac{1}{1.4}}\right)\left(1 - \frac{2 \times \frac{1}{2}}{\frac{1}{2} + \frac{1}{1.8} + \frac{1}{1.4}} \times \frac{1}{2}\right) = 0.31209$$

(ii) $\mu' \{(m_1, w_2), (m_3, w_3)\}$:

$$U_{m_1}(\mu') = \left(1 - \frac{\frac{1}{1.9}}{\frac{1}{1.9} + \frac{1}{1.4}}\right)\left(1 - \frac{\frac{1}{1.9}}{\frac{1}{1.9} + \frac{1}{1.4}} \times \frac{1}{1.9}\right) = 0.44720$$

Thus, agent m_1 is better off by dumping his higher ability partner for an inferior partner. A similar deviation blocks any other fully matched matching, and if agents are not fully matched in matching μ, then μ is blocked by an unmatched pair. Thus, there is no pairwise stable matching in this example.□

The problem underlying this example is that players prefer to have a smaller number of rival pairs, and the best player would rather have a weaker partner if the number of rival pairs goes down. However, since single players cannot participate in the competition, resulting in receiving the lowest payoffs, it does not make sense to expect that they will stay singles. If the single players becomes a pair, the number of rivals do not change, undermining the motivation for the best player to seek a lower ability partner. Using the second effectiveness function $\rightrightarrows_S$ allows us to define the following alternative stability concept. A matching μ is **pairwise stable via swapping** if and only if (i) $\mu R_m \mu'$ or $\mu R_w \mu'$ for any pairwise deviations $(m, w) \in M \times W$ with $\mu \rightrightarrows_{(m,w)} \mu'$. In the following, we will show that the assortative matching μ^* is uniquely stable in the above sense. We first prove the following lemma, which demonstrates that an assortative swapping improves higher ability players' payoffs.

Lemma 1. *Let μ, $m_\ell, m_k \in M$ with $\ell < k$ (thus $a_{m_\ell} > a_{m_k}$), and $\mu(m_\ell), \mu(m_k) \in W$ with $a_{\mu(m_\ell)} < a_{\mu(m_k)}$. Let μ' be such that $\mu'(m_\ell) = \mu(m_k)$ and $\mu'(m_k) = \mu(m_\ell)$ with $\mu'(x) = \mu(x)$ for all other x by swapping the partners among these two pairs. Then, we have (i) $\sum_{j=1}^{n} \frac{1}{A_j(\mu')} > \sum_{j=1}^{n} \frac{1}{A_j(\mu)}$, and (ii) $\mu' P_{m_\ell} \mu$ and $\mu' P_{\mu'(m_\ell)} \mu$.*

This lemma implies that under any matching $\mu \in \mathcal{M}^F$ with $\mu(m_1) \neq w_1$, the highest ability pair (m_1, w_1) is better off by deviating from μ via swapping. Similarly, under any matching $\mu \in \mathcal{M}^F$ with $\mu(m_1) = w_1$ and $\mu(m_2) \neq w_2$, the next highest ability pair (m_2, w_2) is better off by deviating from μ via swapping, and so on. If a matching $\mu \notin \mathcal{M}^F$, then any unpaired players deviate from μ by becoming a pair. Thus, the following proposition is a trivial consequence in the pairs competition model.

Proposition 2. *In the pairs competition model, the assortative matching μ^* is unique pairwise stable matching via swapping.*

4 Stable Assignments in Oligopolistic Joint Ventures

Here, we provide another matching model with externalities. Consider one-to-one joint ventures between marketing companies $m_i \in M$ and technology companies $w_j \in W$. Given a matching $\mu \in \mathcal{M}$, the pairs of companies form joint ventures, which we call firms, competing in a differentiated-good oligopoly market. Each firm i (similar to the pairs competition, we index firms by the associated marketing company's identity): firm i is $(m_i, \mu(m_i)))$ develops its product i by using the two companies' input. We assume that the marginal cost of production by joint venture (m_i, w_j) is a function of the member companies' abilities:

$$c_i \equiv c(m_i, w_j) = f(a_{m_i}, a_{w_j}),$$

where $\frac{\partial f}{\partial a_m} < 0$, $\frac{\partial f}{\partial a_w} < 0$, and $\frac{\partial^2 f}{\partial a_m \partial a_w} \leq 0$ (submodularity in marginal cost or complementary in ability).

Now, we borrow the model from Shubik (1984) to describe our oligopolistic market.[7] Suppose that there are n products produced by n *active* joint ventures together with a numeraire commodity (the 0th commodity). There is unit mass of identical consumers, each with a quadratic utility function:

$$u = \alpha \sum_{i=1}^{n} x_i - \frac{1}{2} \sum_{i=1}^{n} x_i^2 - \frac{\delta}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} x_k x_j + x_0, \tag{5}$$

where $\delta \in [0, 1)$ is a substitution parameter between products. As δ increases, substitutability increases, and in the limit ($\delta = 1$), the model converges to a homogenous good model. Consumers have income I with numeraire (commodity 0), and let us assume $p_0 = 1$ for simplicity. The budget constraint is:

$$\sum_{i=1}^{n} p_i x_i + x_0 = I.$$

With this utility function, we have the following linear demand function.

Lemma 2. *With quadratic utility function (5), the market demand function for good i is:*

$$x_i(p_i, P) = \frac{\alpha}{1 + n\delta} + \frac{\delta}{1 + n\delta} P - p_i,$$

where $P = \sum_{j=1}^{n} p_j$.

Note that if $\delta = 0$, then demand function is not affected by other firms' prices. Also note that firm i is only competing against the average price (or the prevailing market price), making our analysis much simpler.

Letting joint venture firm i's marginal cost be $c_i > 0$, firm i's profit function is written as:

$$y_i(p_i, P) = (p_i - c_i) x_i(p_i, P).$$

We can calculate the oligopolistic market equilibrium as follows. Here, we assume $\mu \in \mathcal{M}^F$ for simplicity, but even if there are unmatched companies, the results continue to hold by replacing n by $n(\mu)$. We can assure that a joint venture always earns a positive profit by assuming the following regularity condition.

[7] Chen (2019) considered oligopolistic joint ventures with Cournot competition. Here we use Shubik's quadratic utility model since we can parametrize the interdependence of demand of each good by $d \geq 0$.

Regularity Condition 2. *Under the assortative matching μ^*, joint venture (m_n, w_n) obtains a positive profit: i.e.,*

$$\frac{\alpha}{2+(n-1)\delta}+\frac{\delta(1+(n-1)\delta)}{(2+(2n-1)\delta)(2+(n-1)\delta)}\sum_{i=1}^{n}c(m_i,w_i)-\frac{1+n\delta}{2+(2n-1)\delta}c(m_n,w_n)>0.$$

Proposition 3. *Suppose that Regularity Condition 2 holds. For any $\mu \in \mathcal{M}^F$, there exists a unique equilibrium in the oligopolistic joint ventures. Firm i's profit is*

$$y_i(\mu)=\frac{1+(n-1)\delta}{1+n\delta}\left(\frac{\alpha}{2+(n-1)\delta}+\frac{\delta(1+(n-1)\delta)}{(2+(2n-1)\delta)(2+(n-1)\delta)}\sum_{i'=1}^{n}c_{i'}-\frac{1+n\delta}{2+(2n-1)\delta}c_i\right)^2.$$

Moreover, the average equilibrium price is

$$\frac{P}{n}=\frac{\alpha}{2+(n-1)\delta}+\frac{1+(n-1)\delta}{2+(n-1)\delta}\times\frac{\sum_{i'=1}^{n}c_{i'}}{n}.$$

Shapley and Shubik (1972) considered a TU characteristic function form game based on an $n \times n$ value matrix for every possible pair. Here, although a joint venture's profit will be split between the constituent marketing and technology companies, the other ventures formed also matter as they also affect profits. Let $\mu \in \mathcal{M}^F$, and let $v(m, w; \mu) \in \mathbb{R}_+$ be

$$v(m,w;\mu)=\begin{cases}\frac{1+(n-1)\delta}{1+n\delta}\times\left(\frac{\alpha}{2+(n-1)\delta}\right. \\ \left.+\frac{\delta(1+(n-1)\delta)}{(2+(2n-1)\delta)(2+(n-1)\delta)}\sum_{i=1}^{n}c(m_i,\mu(m_i))-\frac{1+n\delta}{2+(2n-1)\delta}c(m,w)\right)^2, & \text{if } \mu(m)=w \\ 0, & \text{otherwise.}\end{cases}$$

We extend Shapley and Shubik's (1972) pairwise stable assignment to our model with externalities. Let a **feasible allocation** (μ, r, s) be a triple of a matching, payoff vectors $r = (r_1, ..., r_m) \in \mathbb{R}^m_+$ and $s = (s_1, ..., s_n) \in \mathbb{R}^n_+$ for $(m_1, ..., m_n)$ and $(w_1, ..., w_n)$, respectively, such that $v(m_i, w_j, \mu) = r_i + s_j$ for all $i, j = 1, ..., n$ with $\mu(m_i) = w_j$, and $r_i = 0$ and $s_j = 0$ for all $m_i = \mu(m_i)$ and $w_j = \mu(w_j)$. A feasible allocation of an assignment problem is **pairwise stable assignment via swapping** if for any $m_k \in M$ and for any $w_\ell \in W$, $r_k + s_\ell \geq v(m_k, w_\ell, \mu')$ holds for μ' induced by a pairwise deviation via swapping. If an assignment (μ, r^*, s^*) is pairwise stable via swapping and if $s^* \leq s'$ holds for any pairwise stable outcome via swapping (μ, r, s), then (μ, r^*, s^*) is said the M**-optimal pairwise stable assignment**. Similarly, we can define the W-optimal pairwise stable assignment. The next lemma is a trivial consequence of submodularity of marginal cost function $f(a_m, a_w)$.

Lemma 3 (strict supermodularity of v). *In the oligopolistic joint ventures model, let m, m', w, and w' be such that $a_m > a_{m'}$ and $a_w > a_{w'}$, and let matchings μ and μ' be $\mu(m) = w'$, $\mu(m') = w$, $\mu'(m) = w$, $\mu'(m') = w'$, and $\mu(x) = \mu'(x)$ for all $x \neq m, m', w, w'$: i.e., μ' is a matching obtained as $\mu \rightrightarrows_{(m,w)} \mu'$. Then, we have $v(m, w; \mu') + v(m', w'; \mu') > v(m, w'; \mu) + v(m', w; \mu)$.*

This supermodularity of v has a simple but important implication—if there is a pairwise stable assignment, its matching must be the assortative μ^*.

Proposition 4. *In an oligopolistic joint venture model, suppose that (μ, π, s) is a pairwise stable assignment via swapping. Then, $\mu = \mu^*$.*

A stable assignment in the oligopolistic joint venture assignment problem can be described by using an output matrix as in Shapley and Shubik (1972). The output for all pairwise deviations from the complete assortative matching μ^* are described in the following $n \times n$ local output matrix around μ^*:

$$X(\mu^*) = \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{pmatrix} = \begin{pmatrix} v(m_1, w_1; \mu^*) & v(m_1, w_2; \mu_{12}) & \cdots & v(m_1, w_n; \mu_{1n}) \\ v(m_2, w_1; \mu_{21}) & v(m_2, w_2; \mu^*) & \cdots & v(m_2, w_n; \mu_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ v(m_n, w_1; \mu_{n1}) & v(m_n, w_2; \mu_{n2}) & \cdots & v(m_n, w_n; \mu^*) \end{pmatrix},$$

where $\mu_{ij} \in \mathcal{M}^F$ denotes a matching induced by swapping m_i, m_j, w_i, w_j from μ^*. Note that $\mu_{ij} = \mu_{ji}$ hold for all $i, j = 1, ..., n$. In this output matrix, the sum of outputs over matched pairs under μ, $\sum_{i=1}^{n} X_{i\mu(i)}$, is maximized at $\mu = \mu^*$. This is consistent with the analysis of Shapley and Shubik (1972). We show a useful lemma for characterizing the extreme points of the set of pairwise stable matchings via swapping.

Lemma 4 (strict increasingness). *$X_{ij} > X_{ij+1}$ for all $i, j = 1, ..., n$ with $j \geq i$, and $X_{ij} > X_{i+1j}$ for all $i, j = 1, ..., n$ with $i \geq j$.*

By Lemmas 3 and 4, we know that X is strictly supermodular and strictly increasing. The following proposition shows that there are pairwise stable assignments, and characterizes the M-optimal pairwise stable matching by using the above output matrix X.[8]

Proposition 5. *In an oligopolistic joint venture model, there exist pairwise stable assignments. Under the M-optimal pairwise stable assignment, the pairwise stable payoff vector for W is minimized at $s^* = (s_1^*, ..., s_n^*)$ where $s_j^* = \sum_{j'=j}^{n-1}(X_{j'+1j'} - X_{j'+1j'+1})$ for any $j \leq n-1$ and $s_n^* = 0$, and the stable payoff vector for M is calculated by $r_j^* = X_{jj} - s_j^*$.*

Bulow and Levin (2006) derived the above simple "minimum competitive salary" formula in the context of firm-worker matching problem with output function $X_{ij} = a_i \times a_j$ (thus with no externalities). Proposition 5 shows that their result can be extended to the problem with externalities as long as supermodularity and increasingness are satisfied.[9]

5 Extentions

Thus far, we have been focusing on cases where heterogeneous ability agents care about either winning probability of their team or monetary payoffs only, and found that our pairwise stability notions have intuitive appeal and nice properties. Here, we provide slight generalizations of the NTU and TU problems.

5.1 Intrinsic Preferences

First, in the pairs competition problem, suppose all $x \in M \cup W$ have the following payoff function: for all $i = 1, ..., n$ with $\mu(m_i) \neq m_i$, and all $x \in \{m_i, w_{\mu(m_i)}\}$,

$$\tilde{U}_x = \epsilon b_x(\mu(x)) + (1 - \epsilon)(\pi_i V - e_x),$$

where $\epsilon \in [0, 1]$, and for all $x \in M \cup W$ with $\mu(x) = x$,

$$\tilde{U}_x = 0.$$

[8] The existence of pairwise stable assignment is shown by constructing the M-optimal pairwise stable assignment. We can characterize the W-optimal pairwise stable matching symmetrically.

[9] See Konishi and Sapozhnikov (2005) for the same result without externalities.

This payoff function is composed of two parts: $b_x(\mu(x))$ incorporates agent x's intrinsic payoff from being matched with $\mu(x)$ independent of externalities or competition outcome. As before, we assume that players decide how much effort to make after a matching μ has been determined. Since the first term enters additively, players' effort decisions depend only on the latter part of $\tilde{U}_x$. Thus, for all $\mu \in \mathcal{M}^F$, equilibrium payoff is

$$\tilde{U}_x(\mu) = \epsilon b_x(\mu(x)) + (1-\epsilon)\, U_x(\mu),$$

where $U_x(\mu)$ is the same as in Section 3:

$$U_x(\mu) = \left[1 - \frac{(n(\mu)-1)\,\frac{1}{A_i(\mu)}}{\sum_{j=1}^{n}\frac{1}{A_j(\mu)}}\right]\left[1 - \frac{(n(\mu)-1)\,\frac{1}{A_i(\mu)}}{\sum_{j=1}^{n}\frac{1}{A_j(\mu)}}\left(\frac{a_x}{A_i(\mu)}\right)^{\frac{\sigma}{1-\sigma}}\right] V.$$

Clearly, when $\epsilon = 0$, this problem degenerates to the pairs competition problem, and to the standard one-to-one matching problem without externalities when $\epsilon = 1$. In the neighborhoods of $\epsilon = 1$ or $\epsilon = 0$, pairwise stable matching via swapping obviously exists. But does this hold when ϵ is significantly far from the end points? Unfortunately, in general, pairwise stable matching via swapping may not exist as we can see from the following example.

Example 2. Consider a figure skating contest with $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Let $\sigma = \frac{1}{2}$, $a_{m_1} = a_{w_1} = 0.5$, $a_{m_2} = a_{w_2} = 0.4$, and $a_{m_3} = a_{w_3} = 0.25$. Personal intrinsic payoffs from the partners are described by the following matrices:

$$\begin{pmatrix} b_{m_1}(w_1) & b_{m_1}(w_2) & b_{m_1}(w_3) \\ b_{m_2}(w_1) & b_{m_2}(w_2) & b_{m_2}(w_3) \\ b_{m_3}(w_1) & b_{m_3}(w_2) & b_{m_3}(w_3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1.14 \end{pmatrix}$$

$$\begin{pmatrix} b_{w_1}(m_1) & b_{w_1}(m_2) & b_{w_1}(m_3) \\ b_{w_2}(m_1) & b_{w_2}(m_2) & b_{w_2}(m_3) \\ b_{w_3}(m_1) & b_{w_3}(m_2) & b_{w_3}(m_3) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1.17 \\ 1 & 1 & 2 \end{pmatrix}$$

Set weight $\epsilon = \frac{1}{2}$. We calculate each player's payoffs from pairs competition under the relevant four matchings:

(i) $\mu^1 = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$: $U_{m_1}(\mu^1) = U_{w_1}(\mu^1) = 0.405$; $U_{m_2}(\mu^1) = U_{w_2}(\mu^1) = 0.291$; $U_{m_3}(\mu^1) = U_{w_3}(\mu^1) = 0.031$.

(ii) $\mu^2 = \{(m_1, w_1), (m_2, w_3), (m_3, w_2)\}$: $U_{m_1}(\mu^2) = U_{w_1}(\mu^2) = 0.384$; $U_{m_2}(\mu^2) = U_{w_2}(\mu^2) = 0.131$; $U_{m_3}(\mu^2) = U_{w_3}(\mu^2) = 0.174$.

(iii) $\mu^3 = \{(m_1, w_3), (m_2, w_1), (m_3, w_2)\}$: $U_{m_1}(\mu^3) = 0.183$; $U_{m_2}(\mu^3) = 0.332$; $U_{m_3}(\mu^3) = 0.160$; $U_{w_1}(\mu^3) = 0.305$; $U_{w_2}(\mu^3) = 0.119$; $U_{w_3}(\mu^3) = 0.257$.

(iv) $\mu^4 = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\}$: $U_{m_1}(\mu^4) = U_{w_1}(\mu^4) = 0.332$; $U_{m_2}(\mu^4) = U_{w_2}(\mu^4) = 0.363$; $U_{m_3}(\mu^4) = U_{w_3}(\mu^4) = 0.277$.

With these numbers, we can show that there is no pairwise stable matching in this example.

- Consider μ^1 and a deviation (m_3, w_2) which leads μ^1 to μ^2 via swapping. We have $\tilde{U}_{m_3}(\mu^1) = \frac{1}{2}(1.14 + 0.031)$, $\tilde{U}_{m_3}(\mu^2) = \frac{1}{2}(1 + 0.174)$, $\tilde{U}_{w_2}(\mu^1) = \frac{1}{2}(1 + 0.291)$, and $\tilde{U}_{w_2}(\mu^2) = \frac{1}{2}(1.17 + 0.131)$. This is a profitable deviation for the deviating pair.

- Consider μ^2 and a deviation (m_1, w_3) which leads μ^2 to μ^3 via swapping. Agent m_1 wants w_3 as long as she is available, and w_3 is happy with m_1 given that m_3 is not available. This is a profitable deviation for the deviating pair.

- Consider μ^3 and a deviation (m_3, w_3) which leads μ^3 to μ^4 via swapping. We have $\tilde{U}_{m_3}(\mu^3) = \frac{1}{2}(1 + 0.160)$, and $\tilde{U}_{m_3}(\mu^4) = \frac{1}{2}(1.14 + 0.277)$, and w_3 is happy to be matched with m_3 whenever possible. This is a profitable deviation for the deviating pair.

- Consider μ^4 and a deviation (m_1, w_1) which leads μ^4 to μ^1 via swapping. This is an assortative swapping, and for the relevant agents intrinsic payoffs from the partners are the same. This is a profitable deviation for the deviating pair.

- Consider any matching with (m_3, w_1). It is easy to see that (m_3, w_3) can deviate from it, since w_3 is always available for m_3.

Thus, there is no pairwise stable matching via swapping in this example.□

Notice that in the above example, all male agents agree that they intrinsically weakly prefer w_3 to w_2, and w_2 to w_1, and all female agents agree that they intrinsically weakly prefer m_3 to

m_2, and m_2 to m_1.[10] Therefore, it is not easy to assure the existence of pairwise stable matching via swapping. If, however, ability ranking agrees with intrinsic preference ranking for all agents, then we have the following result.

Proposition 6. *Suppose that $b_{m_i}(w_1) \geq b_{m_i}(w_2) \geq ... \geq b_{m_i}(w_n)$ for all $m_i \in M$ and $b_{w_i}(m_1) \geq b_{w_i}(m_2) \geq ... \geq b_{w_i}(m_n)$ for all $w_i \in W$. Then, μ^* is the unique pairwise stable matching via swapping.*

5.2 Match Qualities

Here, we introduce match qualities of pairs, and we introduce match qualities in the oligopolistic joint venture problem. Let a match quality matrix Q be

$$Q = \begin{pmatrix} q_{11} & \cdots & q_{1j} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ q_{i1} & \cdots & q_{ij} & \cdots & q_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nj} & \cdots & q_{nn} \end{pmatrix},$$

where $q_{ij} \in (0, 1]$ which captures how good a match between m_i and w_j is. Effectively, this describes how well m_i and w_j can work together.

We incorporate match quality into the oligopolistic joint venture model, assuming that the marginal cost of production for joint venture (m_i, w_j) is

$$c(m_i, w_j) = \frac{1}{q_{ij}} f(a_{m_i}, a_{w_j}).$$

Note that this setup means that if the quality of matches are the same, a high ability pair will have a very efficient production technology. If, however, their match quality q_{ij} is low, production technology can be inefficient (marginal cost is high) even if both m_i and w_j have high abilities. We can define $v(m_i, w_j, \mu)$ in the exactly the same way as above. Matrix Q can take any form based on the match quality of every pair.

[10] In either matrix, payoffs are weakly increasing on every row. It is easy to see that we can modify the example to get the same result with strict preferences.

In this case, the structure of the problem are the same as the original oligopolistic joint ventures, so our solution concept, the pairwise stable assignment via swapping, is well defined. However, with an arbitrary match quality matrix Q, a pairwise stable assignment via swapping may not exist. This is because we can create an arbitrary matrix of the marginal cost of each joint venture, $C = (c(m_i, w_j))_{i,j=1,\dots,n}$ by freely choosing Q. The following example demonstrates this result.

Example 3. Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$ with

$$C = \begin{pmatrix} 0.1 & 0.11 & 0.3 \\ 0.3 & 0.1 & 0.3 \\ 0.3 & 0.3 & 0.3 \end{pmatrix}.$$

There are six full matchings:

(i) $\mu_1(m_1) = w_1, \mu_1(m_2) = w_2, \mu_1(m_3) = w_3$; (m_1, w_2) deviates to create μ_2.

(ii) $\mu_2(m_1) = w_2, \mu_2(m_2) = w_1, \mu_1(m_3) = w_3$; (m_2, w_3) deviates to create μ_3.

(iii) $\mu_3(m_1) = w_2, \mu_3(m_2) = w_3, \mu_3(m_3) = w_1$; (m_1, w_1) deviates to create μ_4.

(iv) $\mu_4(m_1) = w_1, \mu_4(m_2) = w_3, \mu_4(m_3) = w_2$; (m_2, w_2) deviates to create μ_1.

(v) $\mu_5(m_1) = w_3, \mu_5(m_2) = w_1, \mu_5(m_3) = w_2$; (m_1, w_1) deviates to create μ_4.

(vi) $\mu_6(m_1) = w_3, \mu_6(m_2) = w_2, \mu_6(m_3) = w_1$; (m_1, w_1) deviates to create μ_1.

Matching μ_1 is a positively assortative matching, but it is not pairwise stable with swapping due to strong externalities: the deviating pair (m_1, w_2)'s marginal cost slightly increases to 0.11, but the pair (m_2, w_1) has a really high marginal cost 0.3 due to very low quality of match, and the deviators (m_1, w_2) is better off unless δ is very small.

However, if markets are not inter-related ($\delta = 0$), then the problem obviously degenerates to a standard one-to-one matching problem without externalities. Since our pairwise stable matching via swapping also degenerates to the standard pairwise stable matching without externalities, we can ensure a pairwise stable matching by defining preferences with $yR_x y' \Leftrightarrow c(x, y) \leq c(x, y')$. The following proposition is a corollary of Shapley and Shubik (1972).

Proposition 7. *In the oligopolistic competition by joint ventures model, there exists a pairwise stable matching via swapping if $\delta = 0$ (no externalities: local monopoly).*

6 Concluding Remarks

This paper considers stability concepts in one-to-one matching/assignment problems with externalities. We found that the choice of effectiveness functions plays a crucial role in extending the celebrated concept of pairwise stable matching to the environment with externalities. We propose to use pairwise stability with swapping in one-to-one matching problems with externalities. Our extension section shows that it may not easy to assure the existence of pairwise stable matching via swapping, but at least in a conceptual level, our stability notion is a reasonable extension of pairwise stability in matching problems without externalities.

Although our focus on this paper is to seek a reasonable adjustment of pairwise stability that is widely used in the matching literature without externalities, there are other solution concepts that may be useful in these matching/assignment models with externalities. Bloch (1996), Ray and Vohra (1999), and Ray (2008) considered dynamic coalition bargaining of partition function form games without and with transfers. They use stationary Markov perfect equilibrium to predict the resulting coalition structure in their noncoopeative coalition formation games. This approach may be worth investigating especially when the problem does not have pairwise stable matching via swapping (such as Examples 2 and 3), since the game's rules require that once a coalition is formed, the members need to commit to it.

Appendix

We collect all proofs here.

Proof of Proposition 1. Substituting this back into (4), recalling $\frac{Y_{-i}}{Y} = 1 - \pi_i$, we obtain

$$\pi_i = 1 - \frac{(n(\mu) - 1)\,\frac{1}{A_i(\mu)}}{\sum_{j=1}^{n} \frac{1}{A_j(\mu)}}.$$

This implies that Y_i is

$$Y_i = \pi_i Y = \left[1 - \frac{(n(\mu)-1)\,\frac{1}{A_i(\mu)}}{\sum_{j=1}^n \frac{1}{A_j(\mu)}}\right]\left[\frac{(n(\mu)-1)}{\sum_{j=1}^n \frac{1}{A_j(\mu)}}\right] V$$

These results lead to the following formulas that are essential for the analysis of stability of team structure. Recalling (3), we obtain

$$\begin{aligned} e_x &= Y_i\left[(1-\pi_i)\frac{1}{Y}\right]^{\frac{1}{1-\sigma}} a_x^{\frac{\sigma}{1-\sigma}} V^{\frac{1}{1-\sigma}} \\ &= \left[1 - \frac{(n(\mu)-1)\,\frac{1}{A_i(\mu)}}{\sum_{j=1}^n \frac{1}{A_j(\mu)}}\right]\left[\frac{(n(\mu)-1)\,V}{\sum_{j=1}^n \frac{1}{A_j(\mu)}}\right]\left[\frac{(n(\mu)-1)\,\frac{1}{A_i(\mu)}}{\sum_{j=1}^n \frac{1}{A_j(\mu)}} \times \frac{\sum_{j=1}^n \frac{1}{A_j(\mu)}}{(n(\mu)-1)\,V}\right]^{\frac{1}{1-\sigma}} a_x^{\frac{\sigma}{1-\sigma}} V^{\frac{1}{1-\sigma}} \\ &= \left[1 - \frac{(n(\mu)-1)\,\frac{1}{A_i(\mu)}}{\sum_{j=1}^n \frac{1}{A_j(\mu)}}\right]\left[\frac{(n(\mu)-1)\,\frac{1}{A_i(\mu)}}{\sum_{j=1}^n \frac{1}{A_j(\mu)}}\right]\left(\frac{a_x}{A_i(\mu)}\right)^{\frac{\sigma}{1-\sigma}} V \end{aligned}$$

This implies that agent i's payoff is written as

$$\begin{aligned} U_x &= \pi_i V - e_x \\ &= \left[1 - \frac{(n(\mu)-1)\,\frac{1}{A_i(\mu)}}{\sum_{j=1}^n \frac{1}{A_j(\mu)}}\right] V - \left[1 - \frac{(n(\mu)-1)\,\frac{1}{A_i(\mu)}}{\sum_{j=1}^n \frac{1}{A_j(\mu)}}\right]\left[\frac{(n(\mu)-1)\,\frac{1}{A_i(\mu)}}{\sum_{j=1}^n \frac{1}{A_j(\mu)}}\right]\left(\frac{a_x}{A_i(\mu)}\right)^{\frac{\sigma}{1-\sigma}} V \\ &= \left[1 - \frac{(n(\mu)-1)\,\frac{1}{A_i(\mu)}}{\sum_{j=1}^n \frac{1}{A_j(\mu)}}\right]\left[1 - \frac{(n(\mu)-1)\,\frac{1}{A_i(\mu)}}{\sum_{j=1}^n \frac{1}{A_j(\mu)}}\left(\frac{a_x}{A_i(\mu)}\right)^{\frac{\sigma}{1-\sigma}}\right] V. \end{aligned}$$

In order to show that Regularity Condition 1 assures $\pi_i > 0$ for all $i = 1, ..., n$, we use Lemma 1 (i) below. Repeatedly applying Lemma 1 (i), it is easy to see $\sum_{j=1}^n \frac{1}{A_j(\mu^*)} \geq \sum_{j=1}^n \frac{1}{A_j(\mu)}$ for all μ. Moreover, $A_n(\mu^*) \leq A_i(\mu)$ for all μ. Thus, if the Regularity condition is satisfied, then $\pi_i(\mu) > 0$ for all μ and all $i = 1, ..., n$. We have completed the proof.□

Proof of Lemma 1. Let $A_j(\mu) \equiv \left(a(m_j)^{\frac{\sigma}{1-\sigma}} + a(\mu(m_j))^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}$ and $\tilde{A}_j(\mu) \equiv a(m_j)^{\frac{\sigma}{1-\sigma}} + a(\mu(m_j))^{\frac{\sigma}{1-\sigma}}$. Clearly, $\tilde{A}_\ell(\mu') > \tilde{A}_\ell(\mu) > \tilde{A}_k(\mu')$, and $\tilde{A}_\ell(\mu') > \tilde{A}_k(\mu) > \tilde{A}_k(\mu')$, which implies $A_\ell(\mu') > A_\ell(\mu) > A_k(\mu')$ and $A_\ell(\mu') > A_k(\mu) > A_k(\mu')$. Let $\tilde{\Delta} = a(m_\ell)^{\frac{\sigma}{1-\sigma}} - a(m_k)^{\frac{\sigma}{1-\sigma}} > 0$. Note that

$$\begin{aligned} \frac{1}{A_\ell(\mu')} + \frac{1}{A_k(\mu')} &= \frac{1}{\left(a(m_\ell)^{\frac{\sigma}{1-\sigma}} + a(\mu(m_k))^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}} + \frac{1}{\left(a(m_k)^{\frac{\sigma}{1-\sigma}} + a(\mu(m_\ell))^{\frac{\sigma}{1-\sigma}}\right)^{\frac{1-\sigma}{\sigma}}} \\ &= \frac{1}{\left(\tilde{A}_\ell(\mu) + \tilde{\Delta}\right)^{\frac{1-\sigma}{\sigma}}} + \frac{1}{\left(\tilde{A}_k(\mu) - \tilde{\Delta}\right)^{\frac{1-\sigma}{\sigma}}}, \end{aligned}$$

Thus,

$$\frac{1}{\left(\tilde{A}_\ell(\mu)+\tilde{\Delta}\right)^{\frac{1-\sigma}{\sigma}}}+\frac{1}{\left(\tilde{A}_k(\mu)-\tilde{\Delta}\right)^{\frac{1-\sigma}{\sigma}}}-\frac{1}{A_\ell(\mu)}-\frac{1}{A_k(\mu)}$$
$$= \left(\tilde{A}_\ell(\mu)+\tilde{\Delta}\right)^{-\frac{1-\sigma}{\sigma}} - A_\ell(\mu)^{-\frac{1-\sigma}{\sigma}} + \left(\tilde{A}_k(\mu)-\tilde{\Delta}\right)^{-\frac{1-\sigma}{\sigma}} - \tilde{A}_k(\mu)^{-\frac{1-\sigma}{\sigma}} > 0,$$

since $f(x)=x^{-\frac{1-\sigma}{\sigma}}$ is a strictly convex function. This implies

$$\frac{1}{A_\ell(\mu')}+\frac{1}{A_k(\mu')} > \frac{1}{A_\ell(\mu)}+\frac{1}{A_k(\mu)},$$

and

$$\sum_{j=1}^{n}\frac{1}{A_j(\mu')} > \sum_{j=1}^{n}\frac{1}{A_j(\mu)}.$$

Combining the last inequality with $A_\ell(\mu') > A_\ell(\mu)$ and $A_\ell(\mu') > A_k(\mu)$, we have

$$U_x(\mu') = \left[1-\frac{(n(\mu')-1)\,\frac{1}{A_\ell(\mu')}}{\sum_{j=1}^{n}\frac{1}{A_j(\mu')}}\right]\left[1-\frac{(n(\mu')-1)\,\frac{1}{A_\ell(\mu')}}{\sum_{j=1}^{n}\frac{1}{A_j(\mu')}}\left(\frac{a_x^\sigma}{A_\ell(\mu')}\right)^{\frac{1}{1-\sigma}}\right]V.$$

for $x=m_\ell$ and $x=\mu'(m_\ell)$ since $n(\mu')=n(\mu)$. Therefore, it is shown that m_ℓ and $\mu'(m_\ell)$ are better off under μ'.■

Proof of Proposition 2. We first show that any matching that is not positively assortative is not pairwise stable via swapping. Suppose that $\mu \in M^F$ and $\mu \neq \mu^*$. Then, there is at least one ℓ such that $\mu(m_\ell) \neq w_\ell$. Find the smallest of such ℓs and name it k. Then, $\mu(m_j)=w_j$ holds for all $j=1,...,k-1$, and $a_{\mu(m_k)} < a_{w_k}$ and $a_{\mu(w_k)} < a_{m_k}$. Consider a deviation by assortative swapping $\mu \rightrightarrows_{(m_k,w_k)} \mu'$. Since $\mu \in M^F$, $\mu' \in M^F$ holds. By Lemma 1, $\mu' P_{m_k} \mu$ and $\mu' P_{w_k} \mu$ hold, and μ is not pairwise stable via swapping as a result. Next, we show that μ^* is pairwise stable via swapping. Suppose not, then there exists a blocking pair (m_j, w_ℓ) and a matching μ' with $\mu^* \rightrightarrows_{(m_j,w_\ell)} \mu'$. Suppose that $\ell > j$. However, from Lemma 1, we have $U_{m_j}(\mu^*) > U_{m_j}(\mu')$. which contradicts to the assumption that (m_j, w_ℓ) is a blocking pair. The argument for $\ell < j$ is similar. Therefore, μ^* is the unique stable matching via swapping. ■

Proof of Lemma 2. Since $\lambda = 1$ in this case, the f.o.c.s are (we assume that consumers have enough income so that $x_0 > 0$):

$$\alpha - x_i - \delta\sum_{j=1}^{n} x_j = p_i.$$

Summing up over commodities produces:

$$n\alpha - \sum_{j=1}^{n} x_j - n\delta \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} p_j,$$

or

$$\sum_{j=1}^{n} x_j = \frac{1}{1+n\delta}(n\alpha - P),$$

where $P = \sum_{j=1}^{n} p_j$. Substituting this back to the f.o.c., we obtain:

$$\alpha - x_i - \frac{\delta}{1+n\delta}(n\alpha - P) = p_i,$$

or

$$x_i = x_i(p_i, P) = \alpha - p_i - \frac{\delta}{1+n\delta}(n\alpha - P).$$

Thus, the market demand function for good i is:

$$x_i(p_i, P) = \frac{\alpha}{1+n\delta} + \frac{\delta}{1+n\delta}P - p_i.$$

We have completed the proof.■

Proof of Proposition 3. The firm i's f.o.c. with respect to p_i is:

$$\left(\frac{\alpha}{1+n\delta} + \frac{\delta}{1+n\delta}P - p_i\right) + (p_i - c_i)\left(\frac{\delta}{1+n\delta} - 1\right) = 0,$$

or

$$\left(1 + \frac{1+(n-1)\,\delta}{1+n\delta}\right) p_i = \frac{\alpha}{1+n\delta} + \frac{\delta}{1+n\delta}P + \frac{1+(n-1)\,\delta}{1+n\delta}c_i$$

or

$$\frac{2+(2n-1)\,\delta}{1+n\delta}p_i = \frac{\alpha}{1+n\delta} + \frac{\delta}{1+n\delta}P + \frac{1+(n-1)\,\delta}{1+n\delta}c_i$$

$$p_i = \frac{\alpha}{2+(2n-1)\,\delta} + \frac{\delta}{2+(2n-1)\,\delta}P + \frac{1+(n-1)\,\delta}{2+(2n-1)\,\delta}c_i$$

Summing them up, we have

$$P = \frac{\alpha n}{2+(2n-1)\,\delta} + \frac{n\delta}{2+(2n-1)\,\delta}P + \frac{1+(n-1)\,\delta}{2+(2n-1)\,\delta}\sum_{j=1}^{n} c_j.$$

Thus,

$$\frac{2+(n-1)\,\delta}{2+(2n-1)\,\delta}P = \frac{\alpha n}{2+(2n-1)\,\delta} + \frac{1+(n-1)\,\delta}{2+(2n-1)\,\delta}\sum_{i=1}^{n} c_i,$$

or

$$P=\frac{\alpha n}{2+(n-1)\delta}+\frac{1+(n-1)\delta}{2+(n-1)\delta}\sum_{j=1}^{n}c_j.$$

Substituting this into the formula for p_i, we obtain

$$\begin{aligned}
p_i &= \frac{\alpha}{2+(2n-1)\delta}+\frac{\delta}{2+(2n-1)\delta}\left(\frac{\alpha n}{2+(n-1)\delta}+\frac{1+(n-1)\delta}{2+(n-1)\delta}\sum_{j=1}^{n}c_j\right)+\frac{1+(n-1)\delta}{2+(2n-1)\delta}c_i\\
&= \frac{\alpha}{2+(2n-1)\delta}\left(1+\frac{n\delta}{2+(n-1)\delta}\right)+\frac{\delta}{2+(2n-1)\delta}\times\frac{1+(n-1)\delta}{2+(n-1)\delta}\sum_{j=1}^{n}c_j+\frac{1+(n-1)\delta}{2+(2n-1)\delta}c_i\\
&= \frac{\alpha}{2+(n-1)\delta}+\frac{\delta(1+(n-1)\delta)}{(2+(2n-1)\delta)(2+(n-1)\delta)}\sum_{j=1}^{n}c_j+\frac{1+(n-1)\delta}{2+(2n-1)\delta}c_i
\end{aligned}$$

Thus, in equilibrium, x_i is

$$\begin{aligned}
x_i &= \frac{\alpha}{1+n\delta}+\frac{\delta}{1+n\delta}P-p_i\\
&= \frac{\alpha}{1+n\delta}+\frac{\delta}{1+n\delta}\left(\frac{\alpha n}{2+(n-1)\delta}+\frac{1+(n-1)\delta}{2+(n-1)\delta}\sum_{j=1}^{n}c_j\right)\\
&\quad -\left(\frac{\alpha}{2+(n-1)\delta}+\frac{\delta(1+(n-1)\delta)}{(2+(2n-1)\delta)(2+(n-1)\delta)}\sum_{j=1}^{n}c_j+\frac{1+(n-1)\delta}{2+(2n-1)\delta}c_i\right)\\
&= \frac{\alpha\{2+(n-1)\delta+n\delta-(1+n\delta)\}}{(1+n\delta)(2+(n-1)\delta)}\\
&\quad +\frac{\delta\{(2+(2n-1)\delta)(1+(n-1)\delta)-(1+n\delta)(1+(n-1)\delta)\}}{(1+n\delta)(2+(2n-1)\delta)(2+(n-1)\delta)}\sum_{j=1}^{n}c_j-\frac{1+(n-1)\delta}{2+(2n-1)\delta}c_i\\
&= \frac{1+(n-1)\delta}{1+n\delta}\left\{\frac{\alpha}{2+(n-1)\delta}+\frac{\delta(1+(n-1)\delta)}{(2+(2n-1)\delta)(2+(n-1)\delta)}\sum_{j=1}^{n}c_j-\frac{1+n\delta}{2+(2n-1)\delta}c_i\right\}.
\end{aligned}$$

Then, firm i's equilibrium profit is

$$y_i(\mu)=\frac{1+(n-1)\delta}{1+n\delta}\left(\frac{\alpha}{2+(n-1)\delta}+\frac{\delta(1+(n-1)\delta)}{(2+(2n-1)\delta)(2+(n-1)\delta)}\sum_{j=1}^{n}c_j-\frac{1+n\delta}{2+(2n-1)\delta}c_i\right)^2.$$

From the submodularity of $f(a_m,a_w)$, $\sum_{i=1}^{n}c(m_i,w_i)=\min_{\mu\in\mathcal{M}^F}\sum_{i=1}^{n}c(m_i,\mu(m_i))$ holds. Thus, Regularity Condition 2 assures that the contents of the parenthesis is positive and $x_i>0$ holds for all $i=1,...,n$.■

Proof of Lemma 3. It is easy to see $f(\alpha_m,a_w)-f(a_{m'},a_w)\leq f(a_m,a_{w'})-f(a_{m'},a_{w'})<0$, since $a_m>a_{m'}$ and $a_w>a_{w'}$, and $\frac{\partial f}{\partial a_m}<0$, $\frac{\partial f}{\partial a_w}<0$, and $\frac{\partial^2 f}{\partial a_m\partial a_w}\leq 0$. Thus, $c(m,w)+c(m',w')\leq$

$c(m,w') + c(m',w)$ holds. By letting $\Delta_w \equiv c(m',w) - c(m,w) > 0$ and $\Delta_{w'} \equiv c(m',w') - c(m,w') > 0$, we have $\Delta_w - \Delta_{w'} \geq 0$. Let $v(m,w;\mu) = B\left(D + FC(\mu) - Gc(m,w)\right)^2$, where $B = \frac{1+(n(\mu)-1)\delta}{1+n(\mu)\delta}$, $D = \frac{\alpha}{2+(n(\mu)-1)\delta}$, $F = \frac{\delta(1+(n(\mu)-1)\delta)}{(2+(2n(\mu)-1)\delta)(2+(n(\mu)-1)\delta)}$, and $G = \frac{1+n(\mu)\delta}{2+(2n(\mu)-1)\delta}$. We can rewrite $v(m,w;\mu) - v(m',w;\mu') + v(m',w';\mu) - v(m,w';\mu')$ in the following:

$$
\begin{aligned}
& B\left(D + F\left(C(\mu') - \Delta_w + \Delta_{w'}\right) - G\left(c(m',w) - \Delta_w\right)\right)^2 - B\left(D + FC(\mu') - Gc(m',w)\right)^2 \\
& + B\left(D + F\left(C(\mu') - \Delta_w + \Delta_{w'}\right) - G\left(c(m,w') + \Delta_{w'}\right)\right)^2 - B\left(D + FC(\mu') - Gc(m,w')\right)^2 \\
= \;& B\left(2D + F\left(2C(\mu') - \Delta_w + \Delta_{w'}\right) - G\left(2c(m',w) - \Delta_w\right)\right)\left(F\left(-\Delta_w + \Delta_{w'}\right) + G\Delta_w\right) \\
& + B\left(2D + F\left(2C(\mu') - \Delta_w + \Delta_{w'}\right) - G\left(2c(m,w') + \Delta_{w'}\right)\right)\left(F\left(-\Delta_w + \Delta_{w'}\right) - G\Delta_{w'}\right) \\
= \;& B\left(2D + F\left(2C(\mu') - \Delta_w + \Delta_{w'}\right) - G\left(c(m',w) + c(m,w)\right)\right)\left(F\left(-\Delta_w + \Delta_{w'}\right) + G\Delta_w\right) \\
& + B\left(2D + F\left(2C(\mu') - \Delta_w + \Delta_{w'}\right) - G\left(c(m',w') + c(m,w')\right)\right)\left(F\left(-\Delta_w + \Delta_{w'}\right) - G\Delta_{w'}\right) \\
= \;& B\left(2D + F\left(2C(\mu') - \Delta_w + \Delta_{w'}\right) - G\left(c(m',w') + c(m,w')\right)\right)\left(F\left(-\Delta_w + \Delta_{w'}\right) + G\left(\Delta_w - \Delta_{w'}\right)\right) \\
& + BG\left(\Delta_{m'} + \Delta_m\right)\left(F\left(-\Delta_w + \Delta_{w'}\right) + G\Delta_w\right) \\
= \;& B\left(2D + F\left(2C(\mu') - \Delta_w + \Delta_{w'}\right) - G\left(c(m',w') + c(m,w')\right)\right)\left(G - F\right)\left(\Delta_w - \Delta_{w'}\right) \\
& + BG\left(\Delta_{m'} + \Delta_m\right)\left(F\Delta_m + \left(G - F\right)\Delta_w\right) \\
> \;& 0,
\end{aligned}
$$

where $\Delta_m = c(m,w') - c(m,w)$ and $\Delta_{m'} = c(m',w') - c(m',w)$. The last inequality holds since (1) $2D + F\left(2C(\mu') - \Delta_w + \Delta_{w'}\right) - G\left(c(m',w') + c(m,w')\right) > 0$ by regularity condition, and (2) $F < \frac{\delta}{2+(2n(\mu)-1)\delta}G < \frac{1}{2}G$ and thus $F < G$. We completed the proof.■

Proof of Proposition 4. Pick an arbitrary matching $\mu \neq \mu^*$. Then, there is at least one assortative swapping with m, m', w, w' from μ such that $a_m > a_{m'}$ and $a_w > a_{w'}$, and $\mu(m) = w'$ and $\mu(w) = m'$. By feasibility, $\pi_m + s_{w'} = v(m,w';\mu)$ and $\pi_{m'} + s_w = v(m',w;\mu)$ hold, but by Lemma 5, we know $v(m,w;\mu') + v(m',w';\mu') > v(m,w';\mu) + v(m',w;\mu)$ where μ' is obtained by swapping with m, m', w, w'. Thus, there is at least one improving deviation either by (m,w) or (m',w') from (μ, π, s), and no matching other than μ^* can be pairwise stable.□

Proof of Lemma 4. We compare $X_{ij} = v(m_i, w_j; \mu_{ij})$ and $X_{ij+1} = v(m_i, w_{j+1}; \mu_{ij+1})$. Since $C(\mu_{ij}) = C(\mu^*) - c(m_i,w_i) - c(m_j,w_j) + c(m_i,w_j) + c(m_j,w_i)$ and $C(\mu_{ij+1}) = C(\mu^*) - c(m_i,w_i) -$

$c(m_{j+1}, w_{j+1}) + c(m_i, w_{j+1}) + c(m_{j+1}, w_i)$, we have

$$v(m_i, w_j; \mu_{ij}) = B\left[D + F\left(C(\mu^*) - c(m_i, w_i) - c(m_j, w_j) + c(m_i, w_j) + c(m_j, w_i)\right) - Gc(m_i, w_j)\right]^2$$

and

$$v(m_i, w_{j+1}; \mu_{ij+1}).$$
$$= B\left[D + F\left(C(\mu^*) - c(m_i, w_i) - c(m_{j+1}, w_{j+1}) + c(m_i, w_{j+1}) + c(m_{j+1}, w_i)\right) - Gc(m_i, w_{j+1})\right]^2$$

Since the contents of the brackets are positive, $v(m_i, w_j; \mu_{ij}) > v(m_i, w_{j+1}; \mu_{ij+1})$ holds if and only if

$$F\left(-c(m_j, w_j) + c(m_j, w_i)\right) - (G - F)\, c(m_i, w_j) > F\left(-c(m_{j+1}, w_{j+1}) + c(m_{j+1}, w_i)\right) - (G - F)\, c(m_i, w_{j+1}).$$

We will prove the above inequality. Subtracting the RHS from the LHS, we have

$$F\left(-c(m_j, w_j) + c(m_j, w_i)\right) - (G - F)\, c(m_i, w_j) - F\left(-c(m_{j+1}, w_{j+1}) + c(m_{j+1}, w_i) + (G - F)\, c(m_i, w_{j+1})\right)$$
$$= (G - F)\left(c(m_i, w_{j+1}) - c(m_i, w_j)\right) + F\left[\left(c(m_{j+1}, w_{j+1}) - c(m_{j+1}, w_i)\right) - \left(c(m_j, w_j) - c(m_j, w_i)\right)\right]$$
$$> (G - F)\left(c(m_i, w_{j+1}) - c(m_i, w_j)\right) + F\left[\left(c(m_{j+1}, w_j) - c(m_{j+1}, w_i)\right) - \left(c(m_j, w_j) - c(m_j, w_i)\right)\right]$$
$$\geq 0.$$

We have completed the proof.■

Proof of Proposition 5. Shapley and Shubik (1974) showed that if an outcome of an assignment problem is stable then the assignment matrix associated with it is an optimal assignment. Under strict supermodularity and strict increasingness, the unique optimal assignment of the output matrix X is an assortative matching μ^*. Thus, what is left to show is that the minimum stable payoff vector for W that supports this assignment is s^* where $s^*_j = \sum_{j'=j}^{n-1}(X_{j'+1j'} - X_{j'+1j'+1})$ for any $j \leq n-1$ and $s^*_n = 0$.

Suppose not. Then there is a pairwise stable assignment that leaves payoff vector s' for W with $s'_j < s^*_j$ for some $j \leq m$. Obviously, such j must belong to $\{1, ...n-1\}$, suppose that $s'_{n-1} < s^*_{n-1}$, thus $s'_{n-1} < X_{nn-1} - X_{nn}$. Consider a deviation by a pair (f_n, a_{n-1}). Since $s'_n \geq s^*_n = 0$, $r'_n \leq X_{nn}$. Now, $s'_{n-1} + r'_n < X_{nn-1} - X_{nn} + X_{nn} = X_{nn-1}$. This violates the stability, and contradicts with s' being a competitive salary. Thus $s'_{n-1} \geq s^*_{n-1}$. Suppose that $s'_{n-2} < s^*_{n-2}$, thus

$s'_{n-2} < X_{n-1n-2} - (X_{n-1n-1} - (X_{nn-1} - X_{nn}))$. From the previous step, we know $s'_{n-1} \geq s^*_{n-1}$, and thus $\pi'_{n-1} \leq X_{n-1n-1} - s^*_{n-1} = X_{n-1n-1} - (X_{nn-1} - X_{nn})$. Thus, we have

$$\begin{aligned} s'_{n-2} + r'_{n-1} &< X_{n-1n-2} - (X_{n-1n-1} - (X_{nn-1} - X_{nn})) + X_{n-1n-1} - (X_{nn-1} - X_{nn}) \\ &= X_{n-1n-2}. \end{aligned}$$

This violates the stability, and contradicts with s' being a competitive salary. Thus $s'_{n-2} \geq s^*_{n-2}$. Repeated applications of the same logic conclude that any competitive salary vector s' satisfies $s' \geq s^*$. ■

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