Incentivized Kidney Exchange*

Tayfun Sönmez† M. Utku Ünver‡ M. Bumin Yenmez§

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Abstract

Over the last 15 years, kidney exchange has become a mainstream paradigm to increase transplants. However, compatible pairs do not participate, and the full benefits from exchange can be realized only if they do. We propose incentivizing compatible pairs to participate in exchange by insuring their patients against future renal failure via increased priority in the deceased-donor queue. Efficiency and equity analyses of this scheme are conducted and compared with that of kidney exchange in a new dynamic continuum model. We calibrate the model with US data and quantify substantial gains from adopting incentivized exchange in efficiency and access equity.

Keywords: Market design, organ allocation, kidney exchange, equity, efficiency, compatible pairs
JEL codes: D47, C78

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†Boston College, Department of Economics and Distinguished Research Fellow at Koç University; sonmezt@bc.edu

‡Boston College, Department of Economics and Distinguished Research Fellow at Koç University; unver@bc.edu

§Boston College, Department of Economics; bumin.yenmez@bc.edu

1 Introduction

While transplantation is the best remedy for end-stage renal disease, a severe shortage of transplant kidneys persists worldwide. Kidneys for transplantation can be harvested from deceased or living donors. As of February 2018, more than 280,000 kidney transplants from deceased donors and more than 145,000 from living donors have been performed in the US. The number of willing living donors has been considerably higher than the number of living-donor transplants performed, but intended gifts of a large fraction of potential donors have not materialized due to biological incompatibilities. More than thirty percent of potential living donors are blood-type incompatible, and about five percent are tissue-type incompatible, with their intended recipients. Blood type O patients are especially disadvantaged by these biological barriers because they are blood-type compatible with type O donors only. In contrast, blood type A patients are blood-type compatible with donors of types A and O, blood type B patients are blood-type compatible with donors of types B and O, and blood type AB patients are blood-type compatible with donors of all blood types.[1] The resulting disadvantage to type O patients is mitigated in deceased-donor transplants by a policy that reserves type O kidneys for type O patients, but a similar policy is not possible for living-donor transplants since a living-donor kidney is intended as a gift for a loved one.

Kidney exchange emerged as a transplantation modality over the last 15 years to lessen the prohibitive effects of biological barriers on living-donor transplantation. In its most basic form, a kidney exchange is a swap of donors between two patients who are each incompatible with their own donor but compatible with the other patient's donor. Both donors' intended gifts are materialized through the exchange, providing each patient with a transplant. However, type O patients are again less likely to benefit from this transplantation modality. Consider a type O patient unable to receive a transplant from his blood-type incompatible type A donor. The pair can potentially swap donors with a type A patient who has a type O donor; but being blood-type compatible, these pairs are rare, only arriving when they are tissue-type incompatible. Hence, a large number of "underdemanded" type O patients with type A donors compete for a relatively scarce population of "overdemanded" type A patients with type O donors.[2] Ironically, these pairs with highly sought-after type O donors become available for exchange only because of a tissue-type incompatibility. A biological barrier to transplantation results in an increase in the number of living-donor transplants by facilitating a welfare-increasing utilization of living donors.

Of course the competition for an exchange would not be so unfavorable for type O patients with blood-type incompatible donors if all pairs participated in kidney exchange rather than only incompatible pairs. Indeed, when a clearinghouse for organized kidney exchange was initially proposed, market designers advocated a mechanism where all pairs would participate in exchange, whether they are compatible or not (Roth, Sönmez, and Ünver, 2004). However, since patients with compat-

[1] For the US, 45.6 percent of the population is blood type O, 37.8 percent of the population is blood type A, 12.6 percent of the population is blood type B, and 4 percent of the population is blood type AB.

[2] Based on 2012-2014 data from the three largest kidney-exchange clearinghouses in the US, the percentage of pairs with type O patients was in the range 58.4–60.7 percent and the percentage of pairs with type O donors was in the range 30.8–33 percent (Agarwal et al., 2017).

ible donors are in no need of an exchange for a transplant, the practice of kidney exchange evolved mostly without them. Despite the resulting suboptimal utilization of living donors, no kidney exchange system currently offers any incentives for compatible pairs to participate in exchange. This gap is the motivation of our paper. Our main contribution is the introduction and analysis of an incentive scheme that encourages compatible pairs to participate in kidney exchange. The incentive we propose is in the form of priority in the deceased-donor queue in the event the patient needs a repeat transplant, thus serving as an insurance against a future kidney failure.[3] While our proposed incentive scheme can be offered to all compatible pairs, we analyze a version where the target group is the set of "overdemanded" pairs. These are compatible pairs where the blood types of the donor and the patient differ when either the donor is of type O or the patient is of type AB. For these pairs, the donor has a more highly sought-after blood type than the patient, and their participation in exchange directly results in an additional transplant to the patient of an "underdemanded" pair.

Potential welfare gains of our incentive scheme are considerable. Using data from the US, our numerical analysis in Section 5 suggests that the marginal contribution of incentivized exchange to the number of living-donor transplants can even exceed that of kidney exchange itself: In the absence of kidney exchange, 44.17 percent of patients with living donors fail to receive a transplant from their donors. With kidney exchange, the percentage of unutilized living donors reduces to 33.69 percent. Assuming that half of the target group participate in incentivized exchange, the percentage of unutilized living donors further reduces to 24.01 percent. In the limit, when all compatible pairs of the target group participate, the percentage of unutilized living donors reduces to 14.33 percent.

While the primary role of incentivized exchange is to increase the number of living-donor transplants, it also improves equity in access both for living-donor transplants and deceased-donor transplants. Equity in access is one of the main objectives of the Organ Procurement and Transplantation Network (OPTN), the body which oversees the allocation of transplant organs in the US.[4] In the November 2016 OPTN report on equity in access, patient blood type was identified as one of the three main contributors to inequity in deceased-donor transplantation.[5] Based on this report (and consistent with our numerical analysis in Section 5), patients of blood types O and B are disadvantaged in the US compared to patients of blood types A and AB. Incentivized exchange improves

[3] A living donor already receives priority in the deceased-donor queue in the event of a kidney failure.

[4] Effective March 16, 2000, the US Department of Health and Human Services (HHS) implemented a *Final Rule* establishing a regulatory framework for the structure and operations of the OPTN. The primary goal of the OPTN is "to increase and ensure the effectiveness, efficiency, and equity of organ sharing in the national system of organ allocation," and "to increase the supply of donated organs available for transplantation" (Duda, 2005). Initially, the Final Rule only regulated allocation of deceased-donor organs. Since June 2006, its scope has been extended to include living-donor organs: "Under 42 CFR 121.4(a)(6), the Secretary directs the OPTN to develop policies regarding living organ donors and living organ donor recipients, including policies for the equitable allocation of living-donor organs, in accordance with section 121.8 of the final rule." (See `https://www.federalregister.gov/documents/2006/06/16/E6-9401/response-to-solicitation-on-organ-procurement-and-transplantation-network-optn-living-donor`).

[5] The other two are donor service area and patient PRA, which indicates the likelihood of tissue-type incompatibility for a patient. The primary way to reduce inequity to high PRA patients is increasing the pool size. Hence, incentivized exchange can be expected to contribute to this objective as well. Moreover, incentivizing all compatible pairs rather than only overdemanded-type pairs can have a more pronounced benefit for high PRA patients. The report is available at `https://optn.transplant.hrsa.gov/media/2159/equity_in_access_report_201705.pdf`.

equity in access for living-donor transplantation, mainly by increasing transplants to type O patients through donor exchanges with incentivized pairs. This in turn also improves equity in access for deceased-donor transplantation, since type O patients who benefit from incentivized exchange no longer compete for deceased-donor transplants. Blood types O and B are more heavily represented among minority groups, and hence any disadvantage to patients of these blood types also correlates with an inequity of access across different ethnic backgrounds. As such, incentivized exchange also reduces disparities across racial and ethnic groups. To our knowledge, our proposed policy is the first to enhance both the efficiency and equity of the system.[6]

To analyze the efficiency and equity implications of incentivized exchange, we introduce a new and analytically tractable dynamic large-market model of kidney transplantation.[7] Unlike former models that focus on a single organ-allocation technology, our model can be used to analyze the impact of various technologies and policies that are often used together and that interact with each other. Through our model, we analytically show that, while all primary technologies increase overall access to kidney transplants, living-donor transplantation and kidney exchange also reduce equity in access. In contrast, not only is the overall access to transplants increased under incentivized exchange, equity in access is also improved.

1.1 Literature Review

Kidney exchange was originally proposed by Rapaport (1986) and later formulated and analyzed as a market-design problem by Roth, Sönmez, and Ünver (2004, 2005b, 2007). The idea of including compatible pairs in kidney exchange was initially evaluated by Ross and Woodle (2000) and further explored by Roth, Sönmez, and Ünver (2004, 2005a), Sönmez and Ünver (2014), and Nicolò and Rodríguez-Álvarez (2017) in market-design settings. Although this idea was immediately dismissed by Ross and Woodle (2000) on ethical grounds, it has been receiving wider acceptance in recent years (see, for example, Veatch, 2006, Kranenburg et al., 2006, Gentry et al., 2007, Ratner et al., 2010, Steinberg, 2011, and Ferrari et al., 2017). The proof of concept involving exchanges with compatible pairs is documented in Ratner et al. (2010). That study also reports the results of a survey conducted among compatible patient-donor pairs. The pairs' attitudes toward exchange were largely positive, especially if the patient benefitted from the exchange in some form. From a medical ethics perspective, Veatch (2006) and Steinberg (2011) also advocated for incentives. The

[6] ABO-i deceased-donor allocation policy treats different blood types the same way. Therefore, it can be viewed as a procedurally or ex-ante egalitarian policy. However, because of the interaction of the deceased-donor queue with the other transplantation technologies and because the donor to patient ratio for different blood types are not the same in practice, the waiting times can vary a lot across different blood types. Therefore, this policy results in waiting times that are not equal. Our proposal reduces the difference between the longest and shortest waiting times for different blood type deceased-donor queues in comparison to the regular exchange (see Table 3). Therefore, the incentivized exchange may be better than the regular exchange for a social planner who exhibits ex-post inequality aversion. See Grant et al. (2012) for a study of an ex-post egalitarian social welfare function.

[7] While traditional matching models mostly focus on static, discrete settings, the use of large-market and continuum models has become increasingly common over the last decade, especially in the context of market-design applications. These models include Kojima and Pathak (2009), Che and Kojima (2010), Lee (2017), Azevedo and Budish (2012), Azevedo and Leshno (2016), Kojima, Pathak, and Roth (2013), Liu and Pycia (2013), and Ashlagi and Roth (2014). See also Ünver (2010), Baccara, Lee, and Yariv (2016), Anderson et al. (2017), and Akbarpour, Li, and Oveis-Gharan (2017) for dynamic matching models.

literature also explored providing incentives through exchanging the donor of a compatible pair with a younger or genetically closer donor (see Roth, Sönmez, and Ünver, 2004, Ferrari et al., 2017, and Nicolò and Rodríguez-Álvarez, 2017). Such schemes not only can incentivize a limited number of compatible pairs, they can also deter participation due to uncertain and prolonged waiting times. Our proposal is the first one we are aware of that can globally and ex ante provide incentives to compatible pairs using tools that are already acceptable within the transplantation community.[8]

2 A Dynamic Model of Kidney Transplantation

Consider patients who need a kidney transplant, where each patient has a blood type X $\in$ {O, A, B, AB}. Let $\pi_X > 0$ be the inflow rate of new type X patients; that is $\pi_X dt$ is the measure of type X patients who enter in a small time interval dt. Suppose that the expected lifetime while living with kidney disease is distributed with a continuous and strictly increasing distribution function $F(\cdot)$ on the interval $[0, T]$. Then the measure of blood type X patients who are alive after t years is given by $\pi_X[1 - F(t)]$. In the steady state of this model, when a transplantation option is not present, the total mass of type X patients is $\int_0^T \pi_X[1 - F(t)]dt$.

2.1 Biological Barriers to Kidney Transplantation

The best remedy for kidney failure is transplantation. There are two potential biological barriers for kidney transplantation. A patient must be both blood-type compatible and tissue-type compatible with a potential donor to be able to receive his kidney. Type O donors are blood-type compatible with patients of all four blood-types, type A donors are blood-type compatible with patients of blood types A and AB, type B donors are blood-type compatible with patients of blood types B and AB, and type AB donors are blood-type compatible with patients of only blood type AB. Hence, other things being equal, type O patients are at a disadvantage in finding a blood-type compatible kidney donor. We denote blood-type compatibility through a "donation" relation $\triangleright$ over blood types, such that X $\triangleright$ Y means that type X donors are blood-type compatible with type Y patients.

The second potential biological barrier for kidney transplantation is a tissue-type incompatibility. Transplantation is not possible if the patient has preformed antibodies against the donor DNA. To simplify the exposition in the main text, we assume that the probability of tissue-type incompatibility is uniform at θ between a donor and a random patient where $0 < \theta < 1$.[9] Hence, a patient can receive a kidney transplant from a blood-type compatible donor with probability $(1-\theta)$. For an average patient, $\theta \approx 0.05$ according to latest data from the OPTN.[10]

[8]Indeed, after the initial draft of our paper became available, Veale et al. (2017) reported three uses of a variant of our intertemporal insurance scheme, leading to 25 transplants through chain exchanges. This scheme is utilized as follows: The old living donor of a younger patient, who likely will need a kidney transplant in the future, initiates a chain of exchanges in the present by donating to an incompatible pair. In return, the patient receives priority for a kidney at the end of a similar future chain when his kidney fails. The donor has a short donation window due to her old age, and the insurance scheme helps other pairs to receive transplants through chain exchanges in present, in addition to insuring the potential patient originally paired with the donor.

[9]We relax this assumption in Appendix D, allowing a non-uniform probability of tissue-type incompatibility between a donor and patients of different types. This appendix also provides micro foundations for our results.

[10]This is according to the average calculated panel reactive antibody (CPRA) data of kidney deceased-donor queue

2.2 Deceased-Donor Transplantation

The most common source of transplant kidneys in the US (and in much of the western world) is deceased donors. The United Network for Organ Sharing (UNOS) is the federal contractor in charge of allocating deceased-donor organs in the US, and it uses a point system for kidneys. Since deceased-donor organs perish within a very short period, they are allocated as soon as they are harvested. Two important features of the UNOS deceased-donor kidney allocation system are that the waiting time in the queue is the most significant part of the point system and kidneys are reserved for patients with the same blood type, with the exception of blood-type A kidneys that can also be allocated to blood-type AB patients.

In the medical literature, reserving blood-type X organs to blood-type X patients is referred to as **ABO-identical (ABO-i)** allocation policy. Since blood type AB is relatively rare, ABO-i policy is a good approximation for the allocation of deceased-donor kidneys in the US. And given the strong influence of waiting time in the deceased-donor queue, we will assume that deceased-donor kidneys are allocated with **first-in-first-out (FIFO)** matching technology.

Let δ_X be the inflow rate of type X deceased-donor kidneys. There is a shortage of deceased-donor kidneys in practice, so we assume that $\delta_X < \pi_X$ for each blood type X. When a transplanted kidney eventually fails, the recipient reenters the deceased-donor queue as if he were a new patient. We assume that repeat patients' survival function is the same as the new entrants'. Let ϕ^d be the fraction of the steady-state flow of previous recipients who reenter the deceased-donor queue due to a failure of the transplant. Then $\phi^d \delta_X$ is the steady-state flow of type X repeat patients.

At any time the longest-waiting cohort of type X patients receive the incoming type X deceased-donor kidneys. Let this cohort have arrived $t_X^{\mathbf{d},dec}$ years before the current time. Assuming deceased-donor kidneys are the only source of transplants, at steady state we have

$$[\pi_X + \phi^d \delta_X][1 - F(t_X^{\mathbf{d},dec})] = \delta_X.$$

Hence, the time spent on the type X queue at steady state, or equivalently the transplant waiting time for type X patients, can be found as

$$t_X^{\mathbf{d},dec} = F^{-1}\left(1 - \frac{\delta_X}{\pi_X + \phi^d \delta_X}\right).$$

2.3 Living-Donor Transplantation

Living-donor transplantation is the second major source of transplant kidneys. In 2017, 29 percent of kidney transplants in the US were from living donors. Let λ_X fraction of blood-type X patients have a living donor who immediately becomes available for donation once the patient needs a transplant. Patients with no living donors are referred to as **unpaired patients**. In Section 2.2, we assume that the inflow of patients is higher than the inflow of deceased-donor kidneys for each blood type. In the rest of the paper, we strengthen this assumption and assume that the inflow of unpaired patients is higher than the inflow of deceased-donor kidneys for each blood type. This

registrations from the OPTN. CPRA measures the percentage of the US population against which the patient would have tissue-type incompatibility, retrieved from `http://optn.transplant.hrsa.gov` on 04/05/2018.

assumption easily holds in the US and elsewhere. We assume that each patient has at most one living donor, who is of blood type X with probability $p_X > 0$, and also assume that blood types of the patient and his donor are uncorrelated. Then a blood type X patient with a living donor is (both blood-type and tissue-type) compatible with his donor with probability p^l_X, where

$$p^l_O = (1-\theta)p_O, \qquad p^l_A = (1-\theta)(p_O + p_A),$$
$$p^l_B = (1-\theta)(p_O + p_B), \text{ and} \qquad p^l_{AB} = (1-\theta)(p_O + p_A + p_B + p_{AB}) = (1-\theta).$$

We assume that a patient with a compatible living donor receives the kidney as soon as he needs a transplant without ever entering the deceased-donor queue. Then the flow of blood-type X patients who receive a living-donor transplant from their donors is

$$\mathbf{l}_X = p^l_X \lambda_X \pi_X.$$

Although they last longer than deceased-donor transplants, living-donor transplants can also fail. Let $\phi^l \leq \phi^d$ be the steady-state fraction of living-donor transplant recipients who reenter the deceased-donor queue due to failure of their transplant. We assume that reentrants no longer have a donor upon reentry.

For each blood type X, the availability of living-donor transplantation decreases the flow of new patients to the deceased-donor queue by $\mathbf{l}_X$, but a fraction of that figure, $\phi^l \mathbf{l}_X$, reenter due to failure of living-donor transplants. Therefore, the net steady-state flow of patients entering or reentering the blood-type X deceased-donor queue is given as

$$\pi^{\mathbf{l},dec}_X = \pi_X + \phi^d \delta_X + \phi^l \mathbf{l}_X - \mathbf{l}_X = \pi_X + \phi^d \delta_X - (1-\phi^l) p^l_X \lambda_X \pi_X.$$

Observe that, for each blood type X, the availability of living-donor transplantation reduces the steady-state flow of patients entering the deceased-donor queue by $(1-\phi^l)p^l_X \lambda_X \pi_X$. Hence, the availability of living-donor transplantation uniformly benefits all patient groups. However, when the fraction of patients with living donors is the same for all blood types, the benefit is largest for type AB and smallest for type O patients. Furthermore, since $p_A > p_B$, the benefit is larger for type A patients than for type B patients.

At any time the longest-waiting cohort of type X patients without compatible donors receive the incoming type X deceased-donor kidneys. Let this cohort have arrived $t^{\mathbf{l},dec}_X$ years before the current time. At steady state, we have $\pi^{\mathbf{l},dec}_X [1 - F(t^{\mathbf{l},dec}_X)] = \delta_X$, and therefore the time spent on the type X deceased-donor queue by the receiving cohort can be found as

$$t^{\mathbf{l},dec}_X = F^{-1}\left(1 - \frac{\delta_X}{\pi^{\mathbf{l},dec}_X}\right) = F^{-1}\left(1 - \frac{\delta_X}{\pi_X + \phi^d \delta_X - (1-\phi^l) p^l_X \lambda_X \pi_X}\right) < t^{\mathbf{d},dec}_X.$$

3 Kidney Exchange

While the availability of living donation benefits all patient groups, not all willing living donors are able to donate due to an incompatibility with their intended recipients. Despite this difficulty, an increasing number of patients with incompatible living donors are receiving kidney transplants through an exchange of donors with other incompatible patient-donor pairs.

Formally, a two-way **kidney exchange** matches two "mutually compatible" patient-donor pairs:

the patient of the first pair is compatible with the donor of the second pair, and the patient of the second pair is compatible with the donor of the first pair. Through an exchange of donors, both patients receive a kidney transplant. While patients with compatible donors can also participate in such exchanges, their participation so far has been very limited since they can directly receive a transplant from their own donors. Hence, in this section we will restrict our attention to kidney exchanges between incompatible pairs.

We consider a kidney-exchange program that operates in parallel with the deceased-donor queue. A patient with a compatible donor immediately receives a transplant from his donor without entering either the deceased-donor queue or the kidney-exchange pool. A patient with an incompatible donor, on the other hand, joins both the deceased-donor queue and the kidney-exchange pool. The patient accepts the first available kidney through these two programs.

We refer to a pair with a blood-type X patient and a blood-type Y donor as a type X-Y pair. In real-life applications, there are far fewer type A-O pairs in kidney-exchange pools than their reciprocal type, O-A pairs. The reason is that pairs of the former type are blood-type compatible, so they only join the pool when they are tissue-type incompatible. This is a relatively rare event with $\theta \approx 0.05$. Pairs of the latter type, on the other hand, are blood-type incompatible, and they always join the pool. This motivates:

Assumption 1 *For any two distinct blood types X, Y with* $X \triangleright Y$, $\theta p_X \lambda_Y \pi_Y \leq p_Y \lambda_X \pi_X$.

That is, the inflow of type X-Y pairs (who always join the pool) is at least as much as the inflow of type Y-X pairs (who only join the pool when they are tissue-type incompatible). Since θ is small, this assumption easily holds in practical applications.[11]

To simplify the presentation of our analytical results, we also assume that the inflow of type B-A pairs is at least as much as the inflow of type A-B pairs. This assumption is superfluous and symmetric results hold if the inequality is reversed.

Assumption 2 $p_A \lambda_B \pi_B \geq p_B \lambda_A \pi_A$.

Since there are far fewer type A-O pairs in the pool than their reciprocal type O-A pairs by Assumption 1, it is possible to match every A-O pair as soon as they arrive. While an arriving type A-O pair is tissue-type incompatible with a θ fraction of type O-A pairs in the pool, it is mutually compatible with a much larger fraction $(1 - \theta)$. Similarly, for any two distinct blood types X, Y with $X \triangleright Y$, it is possible to match every type Y-X pair as soon as they arrive. This is also the case for any type A-B pair by Assumption 2. It turns out that this simple procedure is an optimal exchange mechanism.

Theorem 1 (ABO-identical exchange is optimal) *Suppose Assumptions 1 and 2 hold. Then the exchange policy where an arriving incompatible pair is immediately matched with a mutually compatible pair of its reciprocal type maximizes the measure of transplants to pairs arriving at that*

[11] Based on 2012-2014 data from the three largest kidney-exchange clearinghouses in the US, the percentage of "underdemanded" O-A, O-B, O-AB, A-AB, B-AB pairs was in the range 41.9-43.1 percent and the percentage of "overdemanded" A-O, B-O, AB-O, AB-A, AB-B pairs was in the range 14-15.2 percent (Agarwal et al., 2017).

instance. Moreover, this policy maximizes the mass of pairs who arrive in any interval that can be matched within that interval.

Observe that the structure of the above described optimal exchange can also accommodate FIFO matching where, whenever possible, an arriving type X-Y pair is matched with the longest waiting pair of its reciprocal type Y-X. This is the kidney-exchange mechanism we consider in this paper.

3.1 Effect of Kidney Exchange on Patients with Living Donors

The following grouping of transplants from kidney exchange is not only helpful to explain the effect of exchange on patients with living donors, but also to highlight the welfare loss that results from excluding compatible pairs from exchange.

1. For each blood type X, transplants due to tissue-type incompatible pairs of type X-X: While these patients are blood-type compatible with their donors, they are tissue-type incompatible. Kidney exchange renders tissue-type incompatibility immaterial for them, since each one can be matched with a mutually compatible pair of identical type as soon as they join the kidney-exchange pool. The net increase in the flow of transplants at steady state is $\theta p_X \lambda_X \pi_X$ for each blood type X.
2. For each pair of distinct blood types X, Y with X $\triangleright$ Y, transplants due to tissue-type incompatible pairs of type Y-X: Tissue-type incompatibility becomes immaterial for such patients as well, since they too can be matched with a mutually compatible pair as soon as they join the kidney-exchange pool. However, the net increase in the flow of transplants at steady state is $2\theta p_X \lambda_Y \pi_Y$, since each tissue-type incompatible pair of type Y-X also facilitates a transplant for a patient of its reciprocal type X-Y.
3. Transplants due to pairs of types A-B and B-A: For patients of type A-B (which has a lower inflow than type B-A under Assumption 2), both blood-type and tissue-type incompatibility become immaterial; they can immediately be matched with a pair of type B-A. The net increase in the flow of transplants at steady state is $2p_B \lambda_A \pi_A$, since each pair of type A-B also facilitates a transplant for a patient of type B-A.

Intuitively, kidney exchange eliminates tissue-type incompatibility from being a barrier to a living-donor transplantation, and, in doing so, it also facilitates an additional transplant to a patient with a blood-type incompatible donor. In addition, it also facilitates transplants to all patients of type A-B, and as many transplants to patients of type B-A. For pairs with O or AB patients, kidney exchange is directly tied to tissue-type incompatibility. Pairs with blood type AB patients in the kidney-exchange pool join the pool only because they are tissue-type incompatible with their own donor. Pairs with blood type O patients in the pool, on the other hand, can only receive a transplant if a mutually compatible pair of their reciprocal type becomes available for exchange through a tissue-type incompatibility. As a result, the effect of kidney exchange on patient groups of types O and AB is modest compared to its effect on patient groups of types A and B. (See also Theorem 2-Part 2 in Section 4.4.) Indeed, in the absence of tissue-type incompatibility (i.e., for $\theta = 0$), the effect of kidney exchange would be exclusively limited to patients of types A and B.

Let $\mathbf{e}_X$ denote the steady-state flow of type X patients who receive a transplant through kidney exchange. For blood type O and any blood type Y, a flow $\theta p_O \lambda_Y \pi_Y$ of type O-Y pairs are matched with type Y-O pairs. Therefore, $\mathbf{e}_O = \theta p_O(\lambda_O \pi_O + \lambda_A \pi_A + \lambda_B \pi_B + \lambda_{AB} \pi_{AB})$. Similarly,

$$\begin{aligned}
\mathbf{e}_A &= \theta p_A(\lambda_A \pi_A + \lambda_{AB} \pi_{AB}) + \theta p_O \lambda_A \pi_A + p_B \lambda_A \pi_A, \\
\mathbf{e}_B &= \theta p_B(\lambda_B \pi_B + \lambda_{AB} \pi_{AB}) + \theta p_O \lambda_B \pi_B + p_B \lambda_A \pi_A, \text{ and} \\
\mathbf{e}_{AB} &= \theta(p_{AB} + p_A + p_B + p_O)\lambda_{AB} \pi_{AB} = \theta \lambda_{AB} \pi_{AB}.
\end{aligned}$$

We can summarize the effect of kidney exchange on pairs with living donors as follows:

1. Type A-B & each type X-Y with Y $\triangleright$ X: Each patient of these types either immediately receives a transplant from his own donor or immediately receives a transplant through kidney exchange. In either case, they immediately drop from the deceased-donor queue.
2. Type B-A & each type X-Y with X $\neq$ Y and X $\triangleright$ Y: Patients of these types join both the kidney-exchange pool and the deceased-donor queue. They wait for a transplant, and hence some do not survive.
   (a) For any of these types X-Y, if the wait in the kidney-exchange pool is less than the wait in the blood type X deceased-donor queue, then all surviving pairs of type X-Y receive a transplant through exchange, and none of them receive a transplant from the deceased-donor queue.
   (b) Since all patients of type X-Y receive the first available kidney, the wait in the kidney-exchange pool cannot be more than the wait in the blood type X deceased-donor queue. If the wait for the kidney-exchange pool X-Y is the same as the blood type X deceased-donor queue, then patients of type X-Y pool with unpaired patients of blood type X. Among those who survive, some receive a transplant through exchange and others receive a transplant from the deceased-donor queue.

3.2 Effect of Kidney Exchange on Deceased-Donor Queues

The effect of kidney exchange on the type AB deceased-donor queue is very similar to the effect of living donation, and it is more straightforward than the other blood types. That is because type AB patients who receive a transplant through kidney exchange do so as soon as they become sick without waiting. Indeed, between living donation and kidney exchange, all type AB patients with living donors immediately receive a transplant, completely bypassing the deceased-donor queue where patients without donors wait. Therefore, the availability of kidney exchange along with living donation decreases the flow of incoming patients to the type AB deceased-donor queue by $\mathbf{l}_{AB} + \mathbf{e}_{AB} = \lambda_{AB} \pi_{AB}$, but a fraction of that figure, $\phi^l \lambda_{AB} \pi_{AB}$, reenter due to failure of living-donor transplants. Therefore, the net inflow of patients entering or reentering the type AB deceased-donor queue is given as

$$\pi_{AB}^{\mathbf{e},dec} = \pi_{AB} + \phi^d \delta_{AB} - (1 - \phi^l)\lambda_{AB} \pi_{AB} = (1 - \lambda_{AB})\pi_{AB} + \phi^d \delta_{AB} + \phi^l \lambda_{AB} \pi_{AB}.$$

At steady state, we have $\pi_{AB}^{\mathbf{e},dec}[1 - F(t_{AB}^{\mathbf{e},dec})] = \delta_{AB}$, and, therefore, the time spent on the type AB deceased-donor queue can be found as

$$t_{AB}^{\mathbf{e},dec} = F^{-1}\left(1 - \frac{\delta_{AB}}{\pi_{AB}^{\mathbf{e},dec}}\right) = F^{-1}\left(1 - \frac{\delta_{AB}}{(1 - \lambda_{AB})\pi_{AB} + \phi^d \delta_{AB} + \phi^l \lambda_{AB} \pi_{AB}}\right) < t_{AB}^{\mathbf{l},dec} < t_{AB}^{\mathbf{d},dec}.$$

For the other blood types, the effect of kidney exchange on the deceased-donor queue is more involved. That is because some of the type A, B, or O patients with living donors have to wait for a kidney exchange. And if the wait becomes too long, they are pooled with unpaired patients for deceased-donor kidneys. For blood type A, patients of type A-AB are the only ones who have to wait for a kidney exchange. Patients of types A-B, A-A, and A-O all receive their transplants as soon as they become sick (either from their own donors or through exchange). Therefore, only patients of type A-AB may have to be pooled with unpaired type A patients in the deceased-donor queue, if a transplant through exchange does not become available before a deceased-donor transplant.

The flow of type A-AB to the kidney-exchange pool is $p_{AB}\lambda_A\pi_A$, whereas the flow of their reciprocal type, AB-A, to the kidney-exchange pool is $\theta p_A\lambda_{AB}\pi_{AB}$. Therefore, in the absence of a deceased-donor queue, the steady-state waiting time for kidney exchange is

$$t_{A-AB} = F^{-1}\left(1 - \frac{\theta p_A\lambda_{AB}\pi_{AB}}{p_{AB}\lambda_A\pi_A}\right)$$

for type A-AB. In contrast, the steady-state flow of patients to the type A deceased-donor queue in the absence of patients of type A-AB is

$$\pi_A^u = (1-\lambda_A)\pi_A + \phi^d\delta_A + \phi^l[\mathbf{l}_A + \mathbf{e}_A]$$

and the waiting time at the type A deceased-donor queue is

$$t_A^u = F^{-1}\left(1 - \frac{\delta_A}{\pi_A^u}\right).$$

Therefore, if $t_{A-AB} \leq t_A^u$, then patients of type A-AB exclusively participate in kidney exchange after a wait of t_{A-AB}, whereas unpaired type A patients wait at the deceased-donor queue for a period of t_A^u. If, on the other hand, $t_{A-AB} > t_A^u$, then the two groups are pooled and the waiting time for the type A deceased-donor queue becomes

$$F^{-1}\left(1 - \frac{\delta_A + \theta p_A\lambda_{AB}\pi_{AB}}{\pi_A^u + p_{AB}\lambda_A\pi_A}\right).$$

Observe that the two groups are pooled if and only if

$$\frac{\delta_A}{\pi_A^u} > \frac{\theta p_A\lambda_{AB}\pi_{AB}}{p_{AB}\lambda_A\pi_A},$$

since the supply-to-demand ratio is more favorable for the deceased-donor queue than the kidney-exchange pool. Since θ is very small, the pooling outcome is expected for the US population.

The analysis is similar for types B and O, but the pooling procedure can have one additional step for type B and two additional steps for type O. For the case of type B, patients of types B-AB or B-A may potentially be pooled with unpaired type B patients for the deceased-donor queue. For the case of type O, patients of types O-AB, O-A, or O-B may potentially be pooled with unpaired type O patients for the deceased-donor queue. (See Appendix A.2 and Theorem 4 therein for a detailed analysis.)

4 A New Proposal: Incentivized Exchange

In Section 3, we have seen that kidney exchange increases transplants from living donors. However, exchanges at present are almost exclusively utilized by incompatible pairs, limiting their

welfare impact. To see how the exclusion of compatible pairs affects the gains from exchange, it is helpful to focus on the grouping in Section 3.1.

The critical types of exchanges are those in group 2: For any two distinct blood types X, Y with X $\triangleright$ Y, a tissue-type incompatible Y-X pair exchanges its donor with a X-Y pair. To simplify the discussion, let X=O and Y=A. Through this exchange, the patient of the tissue-type incompatible A-O pair immediately receives a transplant from the donor of the O-A pair. Hence, in the presence of kidney exchange, whether the A-O pair is tissue-type incompatible or not does not affect when or if its patient receives a transplant. But more essentially, this exchange also benefits an A-O pair. In a way, kidney exchange transforms the "misfortune" of the A-O pair (caused by tissue-type incompatibility) to a life-saving opportunity for the O-A pair. Since the A-O pair is blood-type compatible, they would not have participated in exchange in the absence of tissue-type incompatibility. Kidney exchange not only eliminated tissue-type incompatibility as an obstacle for the transplantation, but it also facilitated a transplant for an additional patient. Put differently, from a social-welfare point of view, there is a welfare loss when a blood-type A patient receives a transplant from a blood-type O donor. This welfare loss is avoided due to the combined roles of tissue-type incompatibility together with kidney exchange, and a more efficient utilization of living donors is obtained. But why depend on tissue-type incompatibility to avoid this welfare loss? Any pair of type A-O, whether they are tissue-type incompatible or not, can participate in kidney exchange, facilitating a transplant for an additional patient. The challenge here is that a compatible A-O pair has no reason to participate in exchange.

As our main contribution, we propose incentivizing such compatible pairs to participate in exchange by giving their patient an "insurance" against a potential future failure of his transplant. The insurance takes the form of prioritizing the patient in the deceased-donor queue in the event of a repeat kidney failure. To incentivize their participation in kidney exchange, these **prioritized reentrants** are placed at the top of the deceased-donor queue of their own blood type altering its FIFO structure. Since the welfare gains are due to inclusion of tissue-type compatible pairs of any type Y-X where Y $\neq$ X and X $\triangleright$ Y, we propose that the incentive scheme be provided for these pairs only. For each such pair of type Y-X, let ρ_{Y-X} be the fraction of compatible pairs who are willing to take up the **incentivized exchange** option.

In the absence of incentivized exchange, there is an abundance of O-A pairs compared to A-O pairs. With incentivized exchange and for high values of ρ_{A-O}, this may change. We assume that compatible pairs only take the incentivized-exchange option if they can immediately participate in exchange, assuring that type A-O remains "overdemanded." We use a terminology where the parameter ρ_{A-O} already takes this potential adjustment into consideration. Hence, we assume:

Assumption 3 *For any two distinct blood types X, Y with X $\triangleright$ Y,*

$$[\rho_{Y-X}(1-\theta)+\theta]p_X\lambda_Y\pi_Y \leq p_Y\lambda_X\pi_X.$$

As in the case of kidney exchange, this assumption assures that it is possible to match every Y-X pair at steady state as soon as they arrive for any two distinct blood types X, Y with X $\triangleright$ Y. And moreover, replacing Assumption 3 with Assumption 1 assures that the optimality result of Theorem

1 continues to hold under incentivized exchange. Hence, we again consider the optimal exchange mechanism where an arriving type X-Y pair, whenever possible, is matched with the longest-waiting pair of its reciprocal type, Y-X.

4.1 Effect of Incentivized Exchange on Patients with Living Donors

Since incentivized exchange simply increases the scope of kidney exchange, the analysis in this section parallels the analysis in Section 3.1. Recall that our target group for incentivized exchange is tissue-type compatible pairs of types A-O, B-O, AB-O, AB-A, and AB-B. Consider such a pair of type Y-X that takes the incentivized-exchange option. The patient of this pair could have received a transplant from his own donor, and, hence, his own transplant does not directly increase the total number of transplants. The increase is due to the patient of the reciprocal type X-Y, with whom they engage in exchange. Therefore, at steady state, the number of living transplants increases by one for each incentivized pair in the target group.

Let Y-X be any type targeted for incentivized exchange. The flow of all Y-X pairs is $p_X\lambda_Y\pi_Y$, the flow of tissue-type compatible Y-X pairs is $(1-\theta)p_X\lambda_Y\pi_Y$, and the flow of Y-X pairs who take the incentivized-exchange option is $\rho_{Y-X}(1-\theta)p_X\lambda_Y\pi_Y$.

For each blood type X, let $\mathbf{i}_X$ denote the steady-state flow of the contribution of incentivized exchange on blood-type X living-donor transplants. Living-donor transplants to blood-type A patients with AB donors increases due to incentivized pairs of type AB-A, living-donor transplants to type B patients with AB donors increases due to incentivized pairs of type AB-B, and living-donor transplants to type O patients with blood-type incompatible donors increases due to incentivized pairs of types A-O, B-O, and AB-O. Each type AB patient with a living donor already receives a living-donor transplant under kidney exchange, and, hence, living-donor transplants to type AB patients do not change. Therefore, $\mathbf{i}_{AB} = 0$,

$$\mathbf{i}_A = \rho_{AB-A}(1-\theta)p_A\lambda_{AB}\pi_{AB}, \qquad \mathbf{i}_B = \rho_{AB-B}(1-\theta)p_B\lambda_{AB}\pi_{AB}, \text{ and}$$
$$\mathbf{i}_O = \rho_{A-O}(1-\theta)p_O\lambda_A\pi_A + \rho_{B-O}(1-\theta)p_O\lambda_B\pi_B + \rho_{AB-O}(1-\theta)p_O\lambda_{AB}\pi_{AB}.$$

Since blood type AB is rare, the flow is modest for types AB-A and AB-B. Therefore, while incentivized exchange increases living-donor transplants to patients of blood types A, B, and O, the main increase is for type O patients. We conclude this section by summarizing the effect of incentivized exchange for each patient-donor type.

1. Type A-B and each type X-Y with Y $\triangleright$ X: Patients of these types continue to immediately receive a living-donor transplant under incentivized exchange. For those types targeted for incentivized exchange, patients of pairs who have taken this option also receive a priority increase in the deceased-donor queue for a potential future failure of their transplants.
2. Each type X-Y with X $\neq$ Y and X $\triangleright$ Y: Patients of these types are the primary (but not the only) beneficiaries of incentivized exchange. For any of these types, waiting time for transplantation decreases while the fraction of its patients who receive a living-donor transplant increases. Patients of any of these types X-Y continue to join both the kidney-exchange pool and the deceased-donor queue. However, due to the increased flow of pairs from their reciprocal type

Y-X, the kidney-exchange option becomes more attractive than before with the inclusion of incentivized exchange. Therefore, patients of type X-Y are more likely to exclusively receive a transplant through kidney exchange, and thus less likely to be pooled with unpaired patients at the type X deceased-donor queue.

3. Type B-A: Patients of type B-A continue to join both the kidney-exchange pool and the type B deceased-donor queue. Incentivized exchange does not affect the flow of the patients of type B-A or the flow of the patients of its reciprocal type, A-B. Therefore, patients of type B-A are affected to the extent that they utilize the type B deceased-donor queue. If they are not pooled with unpaired type B patients at the type B deceased-donor queue in the absence of incentivized exchange, their waiting time either stays the same or decreases. If, on the other hand, they are pooled with unpaired type B patients in the absence of incentivized exchange, then their waiting time may increase or decrease depending on the change of the waiting time at the type B deceased-donor queue. If there is a change, it is small.

4.2 Effect of Incentivized Exchange on Deceased-Donor Queues

For each blood type X, let $\mathbf{c}_X$ denote the flow of incentivized compatible pairs with blood-type X patients. Since no pair with a type O patient is incentivized, $\mathbf{c}_O = 0$. For other blood types:

$$\begin{aligned}\mathbf{c}_A =&\rho_{A-O}(1-\theta)p_O\lambda_A\pi_A, \qquad \mathbf{c}_B = \rho_{B-O}(1-\theta)p_O\lambda_B\pi_B, \text{ and}\\ \mathbf{c}_{AB} =&\rho_{AB-O}(1-\theta)p_O\lambda_{AB}\pi_{AB} + \rho_{AB-A}(1-\theta)p_A\lambda_{AB}\pi_{AB} + \rho_{AB-B}(1-\theta)p_B\lambda_{AB}\pi_{AB}.\end{aligned}$$

Derivation of the steady-state waiting times at deceased-donor queues with incentivized exchange closely follows the analysis in Section 3.2 with two simple modifications. For each blood type X, a flow $\phi^l\mathbf{c}_X$ of deceased-donor kidneys are now reserved for prioritized reentrants. Furthermore, the flow of new unpaired patients competing for the remaining flow of deceased-donor kidneys is adjusted by removing the flow $\phi^l\mathbf{c}_X$ of prioritized reentrants. We further assume that $\delta_X > \phi^l\mathbf{c}_X$ so that all prioritized reentrants get matched right away.[12] In the opposite direction, the steady-state flow of new unpaired patients increases by $\phi^l\mathbf{i}_X$ due to failure of the additional living-donor transplants carried out under incentivized exchange.

As in Section 3.2, the derivation of the waiting time at the blood type AB deceased-donor queue is straightforward. The inflow of patients entering or reentering the queue does not change since $\mathbf{i}_{AB} = 0$, but a flow of $\phi^l\mathbf{c}_{AB}$ of them are prioritized, immediately receiving deceased-donor transplants. Therefore, the flow of nonprioritized patients entering or reentering the type AB deceased-donor queue is given as

$$\pi^{\mathbf{i},dec}_{AB} = \pi^{\mathbf{e},dec}_{AB} - \phi^l\mathbf{c}_{AB} = (1-\lambda_{AB})\pi_{AB} + \phi^d\delta_{AB} + \phi^l\lambda_{AB}\pi_{AB} - \phi^l\mathbf{c}_{AB}.$$

Since nonprioritized & unpaired type AB patients compete for a reduced flow, $\delta_{AB} - \phi^l\mathbf{c}_{AB}$, of deceased-donor kidneys, at steady state we have $\pi^{\mathbf{i},dec}_{AB}[1 - F(t^{\mathbf{i},dec}_{AB})] = \delta_{AB} - \phi^l\mathbf{c}_{AB}$, and, therefore,

[12] This inequality easily holds with the estimated US parameters. In particular, $\mathbf{c}_O = 0$ and $\delta_O > 0$, and $\delta_A/\phi^l\mathbf{c}_A = 11.92$, $\delta_B/\phi^l\mathbf{c}_B = 7.69$, and $\delta_{AB}/\phi^l\mathbf{c}_{AB} = 5.36$ when for all eligible $X-Y$, $\rho_{X-Y} = 1$. The same holds for the de-facto deceased-donor flows, (δ'_X), defined in Section 5.

the time spent at the type AB deceased-donor queue can be derived as

$$t_{AB}^{\mathbf{i},dec} = F^{-1}\left(1 - \frac{\delta_{AB} - \phi^l \mathbf{c}_{AB}}{\pi_{AB}^{\mathbf{i},dec}}\right) = F^{-1}\left(1 - \frac{\delta_{AB} - \phi^l \mathbf{c}_{AB}}{\pi_{AB}^{\mathbf{e},dec} - \phi^l \mathbf{c}_{AB}}\right) > t_{AB}^{\mathbf{e},dec}.$$

The waiting time at the deceased-donor queue strictly increases under incentivized exchange for nonprioritized type AB patients. That is expected because, while no additional transplants are carried out for type AB patients, prioritized reentrants receive their deceased-donor transplants immediately without any wait. That means those who have to wait will have to wait longer.

Derivation of the steady-state waiting times at the other deceased-donor queues parallel the analysis in Section 3.2, but due to incentivized pairs of their reciprocal types, a transplant through exchange becomes more likely for patients of types A-AB, B-AB, O-A, O-B, and O-AB. As such, they are less likely to be pooled with unpaired patients with the introduction of incentivized exchange. (See the discussion in Appendix A.2 and Theorem 4 therein for a detailed analysis.)

For patients of blood type A, the main benefit of incentivized exchange is an increased flow of living-donor transplants due to the exchanges with incentivized pairs of type AB-A. The benefit is direct for patients of type A-AB, and indirect through reduced competition for deceased-donor kidneys for nonprioritized & unpaired type A patients. Since blood type AB is rare, the magnitude of this benefit is modest. For nonprioritized & unpaired type A patients, there is also a cost of incentivized exchange in the form of a reduction in their access to deceased-donor kidneys. A flow $\phi^l \mathbf{c}_A$ of type A deceased-donor kidneys are reserved under incentivized exchange for prioritized reentrants of type A. Since ϕ^l is small, this effect is modest as well (especially if ρ_{A-O} is not very high). Due to these opposing effects, the steady-state waiting time at the type A deceased-donor queue may increase or decrease, but the change is expected to be small in either case. Similar arguments also hold for blood type B.

For patients of blood type O, however, the benefits are more profound. Since θ is small, patients of types O-A, O-B, and O-AB are expected to be pooled with unpaired type O patients in the absence of incentivized exchange. The direct benefit of incentivized exchange for patients of types O-A, O-B, and O-AB is an increase in living-donor transplants due to exchanges with incentivized pairs of their reciprocal types. Owing to reduced competition for deceased-donor kidneys, the same effect indirectly benefits unpaired type O patients as well. Unlike other blood types, the lack of prioritized reentrants of type O means that the only potential cost to unpaired blood type O patients is the additional flow $\phi^l \mathbf{i}_O$ of returning patients due to the failure of additional living transplants carried out with incentivized exchange. As such, patients of blood type O are the primary beneficiaries of incentivized exchange.

4.3 Balanced Incentivized Exchange

While access to living-donor transplantation increases with the introduction of incentivized exchange, its benefits are mostly directed to type O patients. This can be considered a desirable feature of incentivized exchange, since the type O patient population is disadvantaged under living-donor transplantation and kidney exchange. One possible case that can be made against incentivized exchange is that not only are its benefits uneven between various patient groups, it is actually

detrimental for some: Nonprioritized & unpaired type AB patients are made worse off, and non-prioritized & unpaired type A or type B patients could be made worse off with the introduction of incentivized exchange. We next introduce a variant of incentivized exchange that does not hurt any patient population.

Consider a compatible type AB-O pair who joins the kidney-exchange pool to benefit from the insurance provided by incentivized exchange. The pair exchanges donors with a type O-AB pair, thus directly helping a type O patient. If, in the future, the incentivized patient needs a repeat transplant due to a subsequent kidney failure, he is given priority in the type AB deceased-donor queue. So participation of the AB-O pair in the pool benefits the type O patient population but incurs a cost on the type AB patient population. Thus, what they bring to the system and what they may take from it are not "balanced." They bring an extra type O living-donor kidney to the system today, but they may take a type AB deceased-donor kidney in the future. If this feature is considered undesirable, there is an easy adjustment. Type AB patient can instead receive priority in the blood type O deceased-donor queue. And in general, the patient of any pair who participates in incentivized exchange can be given priority for the deceased-donor queue of his donor's blood type rather than his own blood type. While this modification changes the ABO-i feature of the deceased-donor queue, it is medically feasible since the pair is blood-type compatible. We refer to this variant of our proposed mechanism as **balanced incentivized exchange**. As we show in the next section, all patient groups weakly benefit from balanced incentivized exchange.

4.4 Equity Implications of Incentivized Exchange

We next present two results which formulate how access to living-donor and deceased-donor transplantation differ across blood types with the introduction of each transplantation modality. For these analytical results, we consider a baseline scenario where no blood type has an advantage over another for access to transplantation beyond the asymmetry induced by blood-type compatibility and the impact of the transplantation modalities analyzed.

The following result formulates how access to living-donor transplantation differs with the successive introduction of living-donor transplantation, kidney exchange, and either version of incentivized exchange.

Theorem 2 *Suppose Assumptions 2 and 3 hold. Let $p_A > p_B$. Suppose that $\lambda_X = \lambda$ for any blood type X, and $\frac{\pi_X}{\pi_Y} = \frac{p_X}{p_Y}$ for any two blood types X and Y. Suppose also that the fraction of pairs taking the incentivized exchange is uniform at a fixed $\rho < 1$ for any eligible type. Then:*

1. *For living donation only, the access to living donation is ranked as*
$$\frac{\mathbf{l}_O}{\pi_O} < \frac{\mathbf{l}_B}{\pi_B} < \frac{\mathbf{l}_A}{\pi_A} < \frac{\mathbf{l}_{AB}}{\pi_{AB}}.$$
2. *Kidney exchange by itself increases access to living-donor transplantation for patients of type B the most, patients of type A the next, and patients of types AB and O equally and last: $\frac{\mathbf{e}_B}{\pi_B} > \frac{\mathbf{e}_A}{\pi_A} > \frac{\mathbf{e}_{AB}}{\pi_{AB}} = \frac{\mathbf{e}_O}{\pi_O}$. With the inclusion of kidney exchange, overall access to living donation is ranked as*
$$\frac{\mathbf{l}_O + \mathbf{e}_O}{\pi_O} < \frac{\mathbf{l}_B + \mathbf{e}_B}{\pi_B} = \frac{\mathbf{l}_A + \mathbf{e}_A}{\pi_A} < \frac{\mathbf{l}_{AB} + \mathbf{e}_{AB}}{\pi_{AB}} = \lambda.$$

3. *Incentivized exchange (or its balanced version) by itself increases access to living-donor transplantation for patients of type O the most, patients of types A and B equally and next, and does not increase access for patients of type AB:* $\frac{\mathbf{i}_O}{\pi_O} > \frac{\mathbf{i}_A}{\pi_A} = \frac{\mathbf{i}_B}{\pi_B} > \frac{\mathbf{i}_{AB}}{\pi_{AB}} = 0$. *With the inclusion of either version of incentivized exchange, overall access to living donation is ranked as*

$$\frac{\mathbf{l}_O + \mathbf{e}_O + \mathbf{i}_O}{\pi_O} < \frac{\mathbf{l}_B + \mathbf{e}_B + \mathbf{i}_B}{\pi_B} = \frac{\mathbf{l}_A + \mathbf{e}_A + \mathbf{i}_B}{\pi_A} < \frac{\mathbf{l}_{AB} + \mathbf{e}_{AB} + \mathbf{i}_{AB}}{\pi_{AB}} = \lambda.$$

Our last result formulates how access to deceased-donor transplantation differs with the successive introduction of deceased-donor transplantation, living-donor transplantation, kidney exchange, and either version of incentivized exchange.

Theorem 3 *Suppose Assumptions 2 and 3 hold. Let* $p_A > p_B$. *Suppose that* $\lambda_X = \lambda$ *for any blood type X, and* $\frac{\delta_X}{\delta_Y} = \frac{\pi_X}{\pi_Y} = \frac{p_X}{p_Y}$ *for any two blood types X and Y. Suppose also that the fraction of pairs taking the incentivized exchange is uniform at a fixed* $\rho < 1$ *for any eligible type. Then:*

1. *With deceased-donor transplantation only, the waiting time at each deceased-donor queue is the same for any blood type X:*

$$t_O^{\mathbf{d},dec} = t_A^{\mathbf{d},dec} = t_B^{\mathbf{d},dec} = t_{AB}^{\mathbf{d},dec}.$$

2. *Introduction of living-donor transplantation reduces the waiting time at each deceased-donor queue. The changes in waiting times and the waiting times are ranked as follows:*

$$(t_{AB}^{\mathbf{d},dec} - t_{AB}^{\mathbf{l},dec}) > (t_A^{\mathbf{d},dec} - t_A^{\mathbf{l},dec}) > (t_B^{\mathbf{d},dec} - t_B^{\mathbf{l},dec}) > (t_O^{\mathbf{d},dec} - t_O^{\mathbf{l},dec}),$$
$$t_{max}^{\mathbf{l},dec} = t_O^{\mathbf{l},dec} > t_B^{\mathbf{l},dec} > t_A^{\mathbf{l},dec} > t_{AB}^{\mathbf{l},dec} = t_{min}^{\mathbf{l},dec}.$$

Further suppose that θ *and* ϕ^l *are sufficiently small. Then:*

3. *Introduction of kidney exchange in addition to living-donor transplantation further reduces the waiting time at each deceased-donor queue, but more for blood type B than blood type A equalizing the deceased-donor queue waiting times for these two blood types. The combination of kidney exchange and living-donor transplantation reduces the waiting time at the type AB deceased-donor queue the most, at the type A and type B deceased-donor queues equally and next, and at the type O deceased-donor queue the least:*

$$(t_{AB}^{\mathbf{d},dec} - t_{AB}^{\mathbf{e},dec}) > (t_A^{\mathbf{d},dec} - t_A^{\mathbf{e},dec}) = (t_B^{\mathbf{d},dec} - t_B^{\mathbf{e},dec}) > (t_O^{\mathbf{d},dec} - t_O^{\mathbf{e},dec}).$$

The inclusion of kidney exchange with living-donor transplantation and deceased-donor transplantation results in the following ranking of the waiting times:

$$t_{max}^{\mathbf{e},dec} = t_O^{\mathbf{e},dec} > t_B^{\mathbf{e},dec} = t_A^{\mathbf{e},dec} > t_{AB}^{\mathbf{e},dec} = t_{min}^{\mathbf{e},dec}.$$

4.(a) *Inclusion of incentivized exchange with kidney exchange, living-donor transplantation, and deceased-donor transplantation decreases the waiting times at the type O, A, and B deceased-donor queue but increases it at the type AB deceased-donor queue. The waits at the type A and B deceased-donor queues continue to be equal:*

$$t_O^{\mathbf{i},dec} < t_O^{\mathbf{e},dec}, \qquad t_A^{\mathbf{i},dec} = t_B^{\mathbf{i},dec} < t_A^{\mathbf{e},dec} = t_B^{\mathbf{e},dec}, \qquad t_{AB}^{\mathbf{i},dec} > t_{AB}^{\mathbf{e},dec}.$$

(b) *Inclusion of balanced incentivized exchange with kidney exchange, living-donor transplantation, and deceased-donor transplantation decreases the waiting times at the type A, B, and*

O deceased-donor queues, while keeping it the same at the type AB deceased-donor queue. While the type A and type B deceased-donor queues continue to have equal waits, the type O deceased-donor queue continues to have the longest wait, whereas the type AB deceased-donor queue continues to have the shortest wait among the four deceased-donor queues.
The difference between the longest and the shortest wait times decreases with the introduction of balanced incentivized exchange:

$$t_O^{\mathbf{b},dec} < t_O^{\mathbf{e},dec}, \qquad t_A^{\mathbf{b},dec} = t_B^{\mathbf{b},dec} < t_A^{\mathbf{e},dec} = t_B^{\mathbf{e},dec}, \qquad t_{AB}^{\mathbf{b},dec} = t_{AB}^{\mathbf{e},dec},$$
$$(\underbrace{t_{max}^{\mathbf{b},dec}}_{=t_O^{\mathbf{b},dec}} - \underbrace{t_{min}^{\mathbf{b},dec}}_{=t_{AB}^{\mathbf{b},dec}}) < (\underbrace{t_{max}^{\mathbf{e},dec}}_{=t_O^{\mathbf{e},dec}} - \underbrace{t_{min}^{\mathbf{e},dec}}_{=t_{AB}^{\mathbf{e},dec}}).$$

5 Numerical Model Predictions

In Section 4, we have shown that when a population is homogeneous in attributes related to becoming a kidney patient or deceased donor and to finding a living donor with respect to different blood types, balanced incentivized exchange not only benefits all patient groups but also makes both deceased donation and living donation more equitable than under regular exchange. In this section, we extend this analysis to the US population by calibrating our model with the US patient and donor characteristics. The US population has heterogenous characteristics among different blood types for becoming a kidney patient and for finding a paired living donor. These numerical calculations also give us predictions regarding waiting times and the number of transplants under various transplantation technologies, including the current policies as well as our proposals.[13]

Calibration Parameters						
	O	**A**	**B**	**AB**		
ABO-i deceased-donor flows (δ_X) =	4982	3922	1225	314	Tissue-type incompatibility prob. θ =	0.0473
De-facto deceased-donor flows (δ'_X) =	4726	3818	1347	554	Reentry fraction of the recipients $\phi^l = \phi^d$ =	25.86%
New patient flows (π_X) =	14693	9983	4466	1162	Incentivized-exchange particip. frac. (ρ) =	25%, 50%, 100%
Paired-donor blood-type prob. (p_X) =	0.456	0.378	0.126	0.040	Survival probability function $1 - F(t)$ =	$0.9427e^{-0.1667t}$
Paired-donor fractions (λ_X) =	43.07%	29.32%	31.74%	21.31%		

Table 1: Calibration parameters for the numerical policy experiments; time unit is one year

We report the calibration parameters for our model in Table 1. We explain in Appendix B how these parameters are obtained. The second row of Table 1, *de-facto deceased-donor flows* (δ'_X), requires some further explanation. Deceased-donation regulations in the US explicitly dictate that type O and type B deceased-donor kidneys are to be transplanted to their respective blood-type patients. However, due to various reasons, type O kidneys are occasionally transplanted to type B patients and less frequently to patients of other blood types (see also Subsection 5.2). Moreover, type AB patients occasionally receive kidneys of other blood types. For these reasons, in addition to the strict ABO-i allocation modality, we calculate our model's predictions as if deceased donors arrived according to this observed transplantation distribution across blood types. This is what we refer to as the de-facto deceased-donor flow for each blood type, and we denote them collectively as

[13]We also run simulations with discrete arrivals using the US population characteristics. These give us similar results as the numerical predictions. The simulations are reported in Appendix C.

(δ'_X). We conduct all of our analysis using this vector in addition to (δ_X), the flows used in ABO-i deceased-donor allocation.

We calculate our model's outcome using these calibration parameters and report outcome variables, such as deceased-donation and living-donation recipient flows, $\mathbf{l}_X, \mathbf{e}_X, \mathbf{i}_X$, for different transplantation technologies (see Table 2). We also calculate the nonprioritized deceased-donor queue waiting time for each blood type (see Table 3) and the transplant waiting time for each blood-type-incompatible pair type (see Table 4) using the formulae that we derived in previous sections. Finally, we report the average waiting times for any type of transplant (see Table 5).

5.1 Welfare Consequences

In terms of overall impact, 34.46 percent of patients receive deceased-donor transplants (measured as a fraction of new entrants, π_X; see the last column in the "Total Transplants" section of Table 2). An additional 20.11 percent receive direct living-donor transplants. An additional 3.78 percent of patients benefit from regular exchange, resulting in 1144 more transplants annually.[14] Our new policy proposal, incentivized exchange, helps an additional 1.75 percent of patients (or about 530 additional patients) per quartile of participation of eligible, compatible pairs. Thus, with full participation, an additional 6.98 percent of patients receive living-donor transplants. The marginal impact of full-scale incentivized exchange is almost twofold that of regular exchange. In this case, all exchange technologies help about 10.76 percent of annual new arrivals to receive transplants in addition to direct living donation. Thus, all living-donation technologies help in aggregate about 90 percent of the number of patients that deceased donation does.

The average waiting time for a nonprioritized deceased-donor transplant decreases from 6.51 years to 5.62 years and then to 5.37 years when direct living donation and regular exchange are introduced, respectively (see the fifth column, entitled "Overall," in Table 3). The waiting time further decreases to 5.21 years and then to 5.05 years when the incentivized-exchange participation rate increases from zero to $\rho = 25\%$ and then to $\rho = 50\%$ (see the tenth and fifteenth columns in Table 3, entitled "Overall"). No further decrease occurs with a further increase of ρ, with the exception of incentivized-exchange treatment with strict ABO-i allocation of deceased-donor kidneys.

As we discussed before in Section 4, nonprioritized & unpaired patients of some individual blood types are the only patient groups that can be adversely affected from incentivized exchange. For example, under the de-facto deceased-donor allocation policy, for type A, type B, and type AB nonprioritized recipients, we observe the following (see Table 3, columns for "incentivized exchange"): The waiting times increase from 4.71, 5.80, and 3.88 years in regular exchange to 4.91, 6.17, and 4.20 years in incentivized exchange with $\rho = 50\%$, respectively. As we articulated in Section 4.3, this negative impact can be neutralized by balanced incentivized exchange. Recall that in this policy proposal, prioritized reentrants receive deceased-donor organs of their previous paired donor's blood type rather than their own. In this case with $\rho = 50\%$, waiting times are 4.74, 5.82, and 3.88

[14] For the external validity of our predictions, we refer to a recent empirical paper by Agarwal et al. (2017), which estimates the potential for annual number of kidney exchanges in the US as at most 1350 transplants using micro-level data including various kinds of exchanges.

Model Outcomes: *Patients Receiving Transplant*											
		O		**A**		**B**		**AB**		**Overall**	
Treatments		*Living-Donor Transplants*									
Living-donor transplantation ($\mathbf{l}_X$)		2749.17	*18.71%*	2325.30	*23.29%*	785.76	*17.59%*	235.93	*20.30%*	6096.17	*20.12%*
Regular exchange ($\mathbf{e}_X + \mathbf{l}_X$)		2984.82	*20.31%*	2813.68	*28.18%*	1194.71	*26.75%*	247.65	*21.31%*	7240.85	*23.89%*
Incentivized	$\rho = 25\%$	3483.52	*23.71%*	2835.97	*28.41%*	1202.14	*26.92%*	247.65	*21.31%*	7769.28	*25.64%*
($\mathbf{e}_X + \mathbf{l}_X + \mathbf{i}_X$)	$\rho = 50\%$	3982.23	*27.10%*	2858.26	*28.63%*	1209.56	*27.08%*	247.65	*21.31%*	8297.71	*27.38%*
	$\rho = 100\%$	4979.65	*33.89%*	2902.85	*29.08%*	1224.42	*27.42%*	247.65	*21.31%*	9354.56	*30.87%*
Treatments	**Dec. Donor A.**	*Deceased-Donor Transplants*									
All except Balanced inc.	ABO-i (δ_X)	4981.85	*33.91%*	3921.51	*39.28%*	1224.57	*27.42%*	314.07	*27.03%*		
	De facto (δ'_X)	4726.00	*32.16%*	3815.00	*38.21%*	1347.00	*30.16%*	554.00	*47.68%*		
Balanced inc. $\rho = 25\%$	ABO-i	4852.86	*33.03%*	3997.96	*40.05%*	1262.47	*28.27%*	328.71	*28.29%*		
	De facto	4597.01	*31.29%*	3891.45	*38.98%*	1384.9	*31.01%*	568.64	*48.94%*	10442.00	*34.46%*
$\rho = 50\%$	ABO-i	4723.87	*32.15%*	4074.41	*40.81%*	1300.36	*29.12%*	343.35	*29.55%*		
	De facto	4468.02	*30.41%*	3967.9	*39.75%*	1422.79	*31.86%*	583.29	*50.20%*		
$\rho = 100\%$	ABO-i	4465.89	*30.39%*	4227.31	*42.35%*	1376.16	*30.81%*	372.64	*32.07%*		
	De facto	4210.05	*28.65%*	4120.79	*41.28%*	1498.58	*33.56%*	612.58	*52.72%*		
Treatments	**Dec. Donor A.**	*Total Transplants*									
Deceased-donor transplantation	ABO-i	4981.85	*33.91%*	3921.51	*39.28%*	1224.57	*27.42%*	314.07	*27.03%*	10442.00	*34.46%*
	De facto	4726.00	*32.16%*	3815.00	*38.21%*	1347.00	*30.16%*	554.00	*47.68%*		
Living-donor transplantation	ABO-i	7731.02	*52.62%*	6246.81	*62.57%*	2010.34	*45.01%*	550.00	*47.33%*	16538.17	*54.57%*
	De facto	7475.17	*50.88%*	6140.30	*61.51%*	2132.76	*47.76%*	789.93	*67.98%*		
Regular Exchange	ABO-i	7966.67	*54.22%*	6735.19	*67.47%*	2419.29	*54.17%*	561.71	*48.34%*	17682.85	*58.35%*
	De facto	7710.82	*52.48%*	6628.68	*66.40%*	2541.71	*56.91%*	801.65	*68.99%*		
Incentivized $\rho = 25\%$	ABO-i	8465.37	*57.62%*	6757.48	*67.69%*	2426.71	*54.34%*	561.71	*48.34%*	18211.28	*60.10%*
	De facto	8209.52	*55.87%*	6650.97	*66.62%*	2549.14	*57.08%*	801.65	*68.99%*		
$\rho = 50\%$	ABO-i	8964.08	*61.01%*	6779.78	*67.91%*	2434.14	*54.50%*	561.71	*48.34%*	18739.71	*61.84%*
	De facto	8708.23	*59.27%*	6673.26	*66.85%*	2556.56	*57.25%*	801.65	*68.99%*		
$\rho = 100\%$	ABO-i	9961.50	*67.80%*	6824.37	*68.36%*	2448.99	*54.84%*	561.71	*48.34%*	19796.56	*65.33%*
	De facto	9705.65	*66.06%*	6717.85	*67.29%*	2571.42	*57.58%*	801.65	*68.99%*		
Balanced inc. $\rho = 25\%$	ABO-i	8336.38	*56.74%*	6833.93	*68.46%*	2464.61	*55.19%*	576.36	*49.60%*	18211.28	*60.10%*
	De facto	8080.54	*55.00%*	6727.42	*67.39%*	2587.03	*57.93%*	816.29	*70.25%*		
$\rho = 50\%$	ABO-i	8706.10	*59.25%*	6932.67	*69.44%*	2509.93	*56.20%*	591.00	*50.86%*	18739.71	*61.84%*
	De facto	8450.26	*57.51%*	6826.16	*68.38%*	2632.36	*58.94%*	830.94	*71.51%*		
$\rho = 100\%$	ABO-i	9445.54	*64.29%*	7130.16	*71.42%*	2600.57	*58.23%*	620.29	*53.38%*	19796.56	*65.33%*
	De facto	9189.69	*62.54%*	7023.65	*70.36%*	2723.00	*60.97%*	860.22	*74.03%*		

Table 2: Model outcomes for the flow of patients receiving transplant (measured in numbers per year) for different patient blood types. The percentages on right of each number are the fractions with respect to the new patient flow (π_X).

Model Outcomes: *Average Time to Nonprioritized Deceased-Donor Transplant*															
Dec.	**O**	**A**	**B**	**AB**	**Overall**	**O**	**A**	**B**	**AB**	**Overall**	**O**	**A**	**B**	**AB**	**Overall**
Donor A.	Deceased-donor transplantation					Incentivized $\rho = 25\%$					Balanced inc. $\rho = 25\%$				
ABO-i	6.64	5.83	7.82	7.90	6.51	5.16	4.70	6.52	7.23	5.20	5.30	4.58	6.33	6.94	5.20
De facto	6.93	5.98	7.28	4.79	6.51	5.41	4.85	5.98	4.04	5.21	5.56	4.72	5.81	3.88	5.19
	Living-donor transplantation					Incentivized $\rho = 50\%$					Balanced inc. $\rho = 50\%$				
ABO-i	5.82	4.81	7.04	6.99	5.62	4.70	4.83	6.73	7.53	5.06	4.97	4.59	6.35	6.94	5.05
De facto	6.11	4.95	6.51	3.92	5.62	4.94	4.91	6.17	4.20	5.05	5.23	4.74	5.82	3.88	5.05
	Regular exchange					Incentivized $\rho = 100\%$					Balanced inc. $\rho = 100\%$				
ABO-i	5.67	4.56	6.32	6.94	5.37	4.37	5.03	7.08	8.18	5.00	5.02	4.54	6.29	6.94	5.05
De facto	5.95	4.71	5.80	3.88	5.37	4.64	5.19	6.48	4.55	5.05	5.34	4.69	5.76	3.88	5.06

Table 3: Model outcomes for deceased-donor waiting time for nonprioritized patients from different blood types (measured in years)

Model Outcomes: *Average Time to Transplant for Blood-Type-Incompatible Pair Types*								
Treatments		**O-A**	**O-B**	**O-AB**	**A-B**	**A-AB**	**B-A**	**B-AB**
Dec.&Living-Donor Trans.		pooled w **O**	pooled w **O**	pooled w **O**	pooled w **A**	pooled w **A**	pooled w **B**	pooled w **B**
Regular exchange		pooled w **O**	pooled w **O**	pooled w **O**	0	pooled w **A**	1.89	pooled w **B**
Incentivized	$\rho = 25\%$	pooled w **O**	pooled w **O**	pooled w **O**	0	pooled w **A**	1.89	pooled w **B**
(with ABO-i	$\rho = 50\%$	pooled w **O**	pooled w **O**/4.78	pooled w **O**	0	pooled w **A**	1.89	pooled w **B**
dec. donor a.)	$\rho = 100\%$	3.14	0.90	pooled w **O** /4.50	0	1.00	1.89	3.24
Incentivized	$\rho = 25\%$	pooled w **O**	pooled w **O**	pooled w **O**	0	pooled w **A**	1.89	pooled w **B**
(with de-facto	$\rho = 50\%$	pooled w **O**	4.78	pooled w **O**	0	4.88	1.89	pooled w **B**
dec. donor a.)	$\rho = 100\%$	3.14	0.90	4.50	0	1.00	1.89	3.24

Table 4: Model outcomes for time to transplant for blood-type-incompatible pair types (measured in years). "Pooled w X" means type X-Y pairs are pooled with type X nonprioritized & unpaired patients. This is true for both incentivized and balanced-incentivized treatments except two cases: Pairs of type O-AB are pooled with nonprioritized & unpaired O patients under ABO-i deceased-donor allocation for the incentivized treatment with $\rho = 100\%$, while this is not true for the balanced-incentivized treatment. The same is true for type O-B when $\rho = 50\%$. This happens as blood-type O nonprioritized deceased-donation recipients wait shorter times in the incentivized treatments. These are denoted in the table in those two cells with a "/" sign. The pooled groups' waiting times are reported in Table 3.

Model Outcomes: *Average Time to Any Type of Transplant*															
Dec.	**O**	**A**	**B**	**AB**	**Overall**	**O**	**A**	**B**	**AB**	**Overall**	**O**	**A**	**B**	**AB**	**Overall**
Donor A.	Deceased-donor transplantation					Incentivized $\rho = 25\%$					Balanced inc. $\rho = 25\%$				
ABO-i	6.64	5.83	7.82	7.90	6.51	3.40	2.69	3.49	3.85	3.16	3.41	2.67	3.50	3.88	3.16
De facto	6.93	5.98	7.28	4.79	6.51	3.51	2.74	3.36	2.72	3.17	3.52	2.72	3.36	2.68	3.17
	Living-donor transplantation					Incentivized $\rho = 50\%$					Balanced inc. $\rho = 50\%$				
ABO-i	3.75	3.02	4.29	3.99	3.55	3.19	2.71	3.50	3.82	3.07	3.22	2.68	3.51	3.88	3.08
De facto	3.86	3.08	4.11	2.75	3.55	3.30	2.72	3.37	2.75	3.08	3.32	2.73	3.37	2.68	3.09
	Regular exchange					Incentivized $\rho = 100\%$					Balanced inc. $\rho = 100\%$				
ABO-i	3.61	2.66	3.49	3.88	3.24	2.71	2.66	3.41	3.72	2.81	2.78	2.61	3.45	3.88	2.84
De facto	3.72	2.71	3.35	2.68	3.24	2.81	2.71	3.30	2.81	2.84	2.86	2.66	3.31	2.68	2.84

Table 5: Model outcomes for average waiting time for any type of transplant for different patient blood types (measured in years).

years for type A, type B, and type AB nonprioritized patients, respectively (Table 3, columns for "balanced incentivized"). Moreover, the benefit to type O nonprioritized patients is still substantial, with a wait of 5.23 years under this treatment, instead of 5.95 years under regular exchange.

5.2 Equity Consequences

Blood-type B patients are disadvantaged even when only deceased-donor transplantation is available; they have the longest waiting time for a transplant (see Table 3). Blood type B is at least twice more common among Asian and African minorities of the US population than among Americans of European descent (see Table 6 of Appendix B). African-Americans are known to be relatively more prone to kidney disease, while the type B deceased-donation rate is not much different from that of other blood types. This explains their prolonged waiting times. Thus, the treatment of type B under our proposed policies, as well as type O patients, bears additional importance in equity considerations.

We summarize our main findings regarding the equity consequences of different transplantation policies as follows:

- De-facto deceased-donor and living-donor transplantation together help type AB patients the most, followed by type A and then by type O and type B. While 67.98 percent of all AB patients and 61.51 percent of all A patients benefit from these two modalities, less than 48 percent of all B patients and 51 percent of all O patients receive transplant (these are the fractions $\frac{\delta'_X+\mathbf{l}_X}{\pi_X}$; see the "Total Transplants" section of Table 2).
- At the margin, type B patients benefit the most from regular exchange. Type A patients benefit next most, while type O and type AB patients benefit the least. While 9.15 percent more of type B patients and 4.89 percent more of type A patients benefit from regular exchange, these fractions are 1.01 percent and 1.60 percent respectively for type AB and type O patients (these are the fractions $\frac{\mathbf{e}_X}{\pi_X}$). The widest inequity gap, the gap between the fractions of type O and type AB patients helped as a result of deceased donation, direct living donation, and regular exchange, is more than 16.5 percent.
- Incentivized exchange with $\rho = 50\%$ helps an additional 6.79 percent of type O patients, which is the main beneficiary group under our proposal. The overall transplant numbers are unaffected for type AB, and the increase is modest for type A and type B. The widest inequity gap, the gap between the fractions of type AB and type B patients benefitting from all transplant modalities, decreases to below 12 percent.
- For nonprioritized deceased-donor recipients, the largest waiting-time gap in the deceased-donor transplantation treatment with de-facto allocation is between types B and AB, as 2.49 years (see Table 3). This gap further increases to 2.59 years with living-donor transplantation treatment. Regular exchange decreases the largest gap to 2.07 years (though for this treatment the largest gap is between types O and AB). Balanced incentivized exchange with $\rho = 50\%$ further decreases the largest gap (which is between types B and AB) to 1.97 years.

Thus, balanced incentivized exchange not only helps all patient groups through more transplants, but it also alleviates the inequities faced in access to deceased- and living-donor transplantation among different patient groups due to medical incompatibilities (as in the case of blood type O) and patient-arrival asymmetries (as in the case of blood type B).

6 Conclusion

Participation of compatible pairs in kidney exchange significantly increases the number of living-donor transplants. We propose incentivizing them to do so by insuring their patient against a repeat kidney failure through priority in the deceased-donor queue. Two key aspects of our proposal are inclusion of compatible pairs in exchange and an adjusted priority ranking in the deceased-donor queue. Our proposal is related to three sparsely practiced variants of kidney exchange. In conclusion, we compare and contrast incentivized exchange with these variants.

An *altruistically unbalanced exchange* involves a kidney exchange between one compatible and one incompatible pair. Ross and Woodle (2000) dismisses these exchanges based on ethical grounds.

The phrase "altruistically unbalanced" reflects their dismissal of the concept. Their concern is potential coercion of compatible pairs who have nothing to gain from exchange. In contrast, exchange is no longer "altruistically unbalanced" under incentivized exchange, since patients of participating pairs are insured against a repeat failure.

Under an *indirect exchange*, the donor of an incompatible pair donates a kidney to the deceased-donor queue in exchange for a priority for his patient in the deceased-donor queue. Hence this variant involves an exchange between an incompatible pair and the deceased-donor queue. Ross and Woodle (2000) object to indirect exchange for blood-type incompatible pairs, but support it for blood-type compatible (but tissue-type incompatible) pairs. Consider a type O patient with a type A donor. Under an indirect exchange, the pair donates a type A kidney to the donor queue in exchange for priority for a type O deceased-donor kidney. That is, they receive priority for a more highly sought-after blood type kidney than the kidney they donate. This is the basis of the Ross and Woodle (2000) objection:

> "The indirect ABO-incompatible exchange does create a new ethical concern because it may increase the vulnerability of O blood group recipients. If mechanisms can be developed to avoid increasing the waiting time for blood group O recipients, we would support the implementation of the indirect ABO-incompatible exchange."

In contrast, they support indirect exchange for blood-type compatible pairs because those pairs either donate the same blood type or a more highly sought-after blood type kidney than the one they are prioritized for. While incentivized exchange is also based on priority in the deceased-donor queue, there are two key differences: First, an incentivized pair donates a kidney of a more highly sought-after blood type than its patient is prioritized for. And indeed, it mainly benefits the blood type O patient population. And second, unlike indirect exchange, the priority is only used if the patient needs a repeat transplant. Both factors are in favor of incentivized exchange based on the above-mentioned ethical considerations.

A *voucher for a chronologically incompatible* pair (Veale et al. (2017)) involves a priority for a (typically young) patient of a pair for a future transplant in exchange for a donation from an older donor today. The donor will be too old to donate when the patient is expected to need a transplant. Observe that this variant is very similar to indirect exchange, and indeed it can be interpreted as an intertemporal version of it. Therefore, the same ethical considerations of Ross and Woodle (2000) apply. That is, the case for these exchanges is stronger when the pair is blood-type compatible than when they are blood-type incompatible. Different from an incentivized exchange or an indirect exchange, the first three of these intertemporal exchanges were organized by the National Kidney Registry, which arranges kidney chains initiated by good-samaritan donors.[15] The older donor starts a chain today, and the younger patient receives priority for a kidney at the end of a chain when he needs a transplant in the future. However, these chains almost never end with a type O kidney, and indeed they likely end with a type AB kidney. Hence, honoring the voucher may require artificially terminating a kidney chain, especially if the patient is of type O. Perhaps motivated by these concerns, Veale et al. (2017) suggest that patients also be prioritized at the deceased-donor

[15]These chains are introduced by Roth et al. (2006), and the proof of concept is documented in Rees et al. (2009).

queue in case the patient cannot be placed at the end of a kidney chain. Conceptually incentivized exchange is similar, but it evades the above-mentioned shortcomings since incentivized pairs are blood-type compatible.

In summary, incentivized exchange harbors all the positive elements of the above variants of kidney exchange without suffering from their shortcomings.

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Appendix A Omitted Proofs and Additional Results on Waiting Times

In this appendix, we provide the omitted proofs and some additional results on waiting times. To do that, we first provide a formal model of tissue-type incompatibility: Each patient has a type, depending on against which subset of HLA tissue proteins he has preformed antibodies. We study the limit as the number of types goes to infinity. First, fix the number of types to a finite k. The probability that a patient is of type i is $m_{i,k} > 0$, so that $\sum_i m_{i,k} = 1$. Let $\theta_{i,k}$ be the tissue-type incompatibility probability between any donor and patient of type i. If a donor is tissue-type compatible with a type i patient, then the donor is tissue-type compatible with all patients of type i. We take the number of types, k, to infinity and make some regularity assumptions on the growth of $m_{i,k}$ and $\theta_{i,k}$ in the limit. See Appendix D for details. These assumptions hold for the special case when $\theta_{i,k} = \theta < 1$ and $m_{i,k} \to 0$ for every patient type i as $k \to \infty$ (Lemma 7). In the current appendix as well as for the results in the main text, we use this special case.

We also define steady states formally: A **state** is defined through the measures of type X-Y pairs who have waited t years in the queue, denoted by $(X - Y, t)$ and type X unpaired patients who have waited t years in the queue, denoted by (X, t) for all blood types X and Y and waiting time t. We say that the population under a given policy of transplantation is at a **steady state** when the measures of all $(X - Y, t)$ and (X, t) are constant through time, i.e., the state does not change over time.

A.1 Optimal Regular and Incentivized Exchange

We first formally categorize pair types in the following three classes. The naming of these classes is based on the comparison of the flows of the type and its reciprocal type for ABO-i optimal exchange (Assumptions 1 and 2). The types X-Y with a weakly lower flow than that of Y-X are overdemanded, while the ones with the weakly higher flow are underdemanded. Remaining types are referred to as self-demanded, as each is matched with itself in ABO-i optimal exchange:

- **Overdemanded Types**: These are pair types X-Y such that Y $\triangleright$ X and Y $\neq$ X and pair type A-B. There are six of these types, A-O, A-B, B-O, AB-O, AB-A and AB-B.
- **Self-demanded Types**: These are pair types X-X. There are four of these types, O-O, A-A, B-B, and AB-AB.
- **Underdemanded Types**: These are pair types X-Y such that X $\triangleright$ Y and X $\neq$ Y and pair type B-A. There are six of these types, O-A, O-B, O-AB, A-AB, B-A, and B-AB.

The following lemma characterizes feasible exchanges. It is useful in the proof of Theorem 1. Similar results also appear in Roth, Sönmez, and Ünver (2007) and Ünver (2010), so we skip its proof.

Lemma 1 (Exchange blood-type feasibility) *An underdemanded-type pair can be matched only with an overdemanded-type pair in an exchange. An overdemanded-type pair can be matched*

with any type pair. A self-demanded-type pair can be matched with a same-type or overdemanded-type pair. In particular, the following results hold:

- *An underdemanded-type O-A (or O-B) pair can be matched only with a pair from overdemanded types A-O (or B-O) or AB-O. An underdemanded-type A-AB (or B-AB) pair can be matched only with a pair from overdemanded types AB-A (or AB-B) or AB-O. An underdemanded-type O-AB pair can be matched only with an overdemanded-type AB-O pair.*
- *An overdemanded A-B (or underdemanded B-A) pair can be matched only with a pair from its reciprocal type B-A (or A-B); or from overdemanded types B-O (or A-O), AB-A (or AB-B), or AB-O.*
- *A self-demanded-type X-X pair can be matched with a same-type pair. Additionally, a type O-O pair can be matched only with a pair from overdemanded types A-O, B-O, or AB-O; a type A-A (or B-B) pair can be matched only with a pair from overdemanded types AB-A (or AB-B) or AB-O; and a type AB-AB pair can be matched only with a pair from overdemanded types AB-A, AB-B, or AB-O.*

Proof of Theorem 1. Suppose Assumptions 1 and 2 hold. Under the proposed policy, by Lemma 6 in Appendix D, all self-demanded-type pairs can be matched with their own-type pairs as soon as they arrive. Similarly, type A-B pairs, which have a weakly lower flow rate than that of type B-A by Assumption 2, can be matched as soon as they arrive with type B-A pairs (Lemma 4 in Appendix D). Hence, under this policy only type B-A pairs will remain in the exchange pool at any point in time. These pairs can only be matched with some overdemanded-type pairs by Lemma 1, as type A-B pairs are already committed to other type B-A pairs.

Next consider underdemanded-type pairs except those of B-A. These are type Y-X pairs such that Y $\neq$ X and Y $\triangleright$ X. By Assumption 1, we have $\theta p_Y \lambda_X \pi_X \leq p_X \lambda_Y \pi_Y$. By Lemma 1, they can only be matched with overdemanded-type pairs. Recall that the flow of each type Y-X pair to the exchange pool is $p_X \lambda_Y \pi_Y$. Their reciprocal type X-Y, which is overdemanded, has flow $\theta p_Y \lambda_X \pi_X \leq p_X \lambda_Y \pi_Y$. Hence, we can match all such overdemanded-type X-Y pairs as soon as they enter the pool with their reciprocal-type pairs (by Lemma 5 in Appendix D). As all overdemanded- and self-demanded-type pairs are matched as soon as they arrive, by Lemma 1, the proposed policy achieves the maximum measure of pairs matched. At steady state, as no incompatible overdemanded-type or self-demanded-type pair waits in the pool (i.e., moreover, get immediately matched and help one additional pair), the maximum mass of possible exchanges is also conducted in this manner in any closed time interval.

On the other hand, if we do not conduct the exchanges immediately whenever they become available but only after some time interval, then some of the patients will not survive. Hence, when we do not conduct the exchanges as soon as possible, we will match a strictly smaller mass of pairs than we would have matched under the proposed policy. ■

A.2 Pooling and Waiting Times under Regular and Incentivized Exchange

Recall that for all incentivized-exchange-eligible pairs, i.e., of all types X-Y such that Y $\triangleright$ X, Y $\neq$ X, and the patient and donor have no tissue-type incompatibility, $\rho_{X-Y} \in [0,1]$ is the fraction that participate in incentivized exchange. Let $\rho = (\rho_{X-Y})_{Y \triangleright X, Y \neq X}$ be the vector of such fractions. We use the terms "regular exchange" and "incentivized exchange with $\rho = 0\%$" interchangeably. To determine the steady-state outcomes, we introduce certain flow rates.[16] For each blood type X and each Y $\neq$ X, let

$$\pi^{\mathbf{i}}_{X-Y} = \begin{cases} [\theta + \rho_{X-Y}(1-\theta)]p_Y \lambda_X \pi_X & \text{if Y } \triangleright \text{ X} \\ p_Y \lambda_X \pi_X & \text{otherwise} \end{cases} \tag{1}$$

refer to the pair-type X-Y flow to the exchange pool. Let the **incentivized pair flow** relevant for blood type X be given by

$$\mathbf{c}_X = \Big(\sum_{Y \,:\, Y \triangleright X \& Y \neq X} \rho_{X-Y}(1-\theta)p_Y \Big) \lambda_X \pi_X. \tag{2}$$

Observe that $\phi^l \mathbf{c}_X$ is the reentry flow of previously incentivized type X patients. These patients will be prioritized in the deceased-donor queue of blood type X and will not wait upon reentry. Thus, the effective flow rate of deceased-donor kidneys for nonprioritized type X patients is $\delta_X - \phi^l \mathbf{c}_X$. We also have

$$\pi^{\mathbf{i}}_X = \underbrace{(1-\lambda_X)\pi_X}_{\text{new unpaired}} + \underbrace{\phi^d \delta_X}_{\text{reentry / deceased}} + \underbrace{\phi^l [\mathbf{l}_X + \mathbf{e}_X + \mathbf{i}_X - \mathbf{c}_X]}_{\text{reentry / all live minus incentivized}} \tag{3}$$

as the total **nonprioritized** & **unpaired type X patient flow**.

We define the following ratios:

1. The ratio of the deceased-donor effective flow for nonprioritized patients to the nonprioritized & unpaired patient flow is
$$r_X = \frac{\delta_X - \phi^l \mathbf{c}_X}{\pi^{\mathbf{i}}_X}. \tag{4}$$
2. For each underdemanded type X-Y except B-A (i.e., X $\neq$ Y and X $\triangleright$ Y), the ratio of the flow of incompatible or incentivized type Y-X pairs to the total flow of type X-Y pairs is
$$r_{X-Y} = \frac{\pi^{\mathbf{i}}_{Y-X}}{\pi^{\mathbf{i}}_{X-Y}} = \frac{[\theta + \rho_{Y-X}(1-\theta)]p_X \lambda_Y \pi_Y}{p_Y \lambda_X \pi_X}.$$
3. For the remaining underdemanded type B-A, the ratio of type A-B flow to type B-A flow is
$$r_{B-A} = \frac{\pi^{\mathbf{i}}_{A-B}}{\pi^{\mathbf{i}}_{B-A}} = \frac{p_B \lambda_A \pi_A}{p_A \lambda_B \pi_B}.$$

The first ratio, r_X, is less than one because of our assumption that there is a shortage of deceased-donor kidneys for unpaired new entrants, i.e., $(1-\lambda_X)\pi_X > \delta_X$. The second ratio, r_{X-Y}, is less than one by Assumption 3. Finally, the last ratio, r_{B-A}, is less than or equal to one by Assumption 2. Ratio r_X would be relevant if we wanted to allocate all type X deceased donors that are reserved

[16] Some of these were previously defined throughout Section 4.

for nonprioritized patients to nonprioritized & unpaired type X patients. For an underdemanded type X-Y, ratio r_{X-Y} would be relevant if type X-Y pairs did not receive any deceased donation but only participated in exchange. In these cases, the waiting time for a deceased-donor transplant for nonprioritized & unpaired type X patients would be

$$t_X = F^{-1}\Big(1 - \frac{\delta_X - \phi^l \mathbf{c}_X}{\pi^{\mathbf{i}}_X}\Big),$$

and the waiting time of underdemanded-type X-Y pairs would be

$$t_{X-Y} = F^{-1}\Big(1 - \frac{\pi^{\mathbf{i}}_{Y-X}}{\pi^{\mathbf{i}}_{X-Y}}\Big).$$

However, underdemanded-type pairs have another option besides exchange. If deceased donors become available earlier than the exchange option, they will receive deceased-donor transplants. As we mentioned in the main text, we assume that patients accept the first donor who is offered to them, either through deceased-donor allocation or exchange. Hence, the patient of a type X-Y pair will never wait for a type Y-X pair for exchange if a deceased donor becomes available first, i.e., if $t_{X-Y} > t_X$. As waiting times are strictly decreasing functions of the r ratios defined above, we need to compare these ratios in an iterative manner to decide whether pairs of one or more underdemanded types will also receive deceased-donor transplants.

To this end, we first define X-Y_1 , ..., X-$Y_{k(X)}$ as the ordered list of underdemanded types according to ascending r_{X-Y} ratios, where we have $k(O) = 3$, $k(B) = 2$, $k(A) = 1$, and $k(AB) = 0$ as the respective numbers of underdemanded pair types whose patients have blood types O, B, A, and AB. We define the following potential pooling ratio for each $\ell = 0, \ldots, k(X)$:

$$r_{X,X-Y_1,\ldots,X-Y_\ell} = \frac{\delta_X - \phi^l \mathbf{c}_X + \pi^{\mathbf{i}}_{Y_1-X} + \ldots + \pi^{\mathbf{i}}_{Y_\ell-X}}{\pi^{\mathbf{i}}_X + \pi^{\mathbf{i}}_{X-Y_1} + \ldots + \pi^{\mathbf{i}}_{X-Y_\ell}}. \tag{5}$$

Exchange technology iterative pooling procedure for unpaired and paired patients:

Fix a blood type X. We iteratively consider the following procedure starting with $\ell = 0$.

Step ℓ: Suppose types X-$Y_1, \ldots,$ X-Y_ℓ have already been deemed to be receiving both deceased-donor and exchange transplants.

- If $r_{X-Y_{\ell+1}} < r_{X,X-Y_1,\ldots,X-Y_\ell}$ then type X-$Y_{\ell+1}$ pairs receive both exchange transplants and deceased-donor transplants, together with the nonprioritized & unpaired type X patients and type X-Y_1,..., X-Y_ℓ pairs. We continue with Step $\ell + 1$.
- If $r_{X-Y_{\ell+1}} \geq r_{X,X-Y_1,\ldots,X-Y_\ell}$ then all type X-$Y_{\ell+1}$,..., X-$Y_{k(X)}$ pairs only receive exchange transplants but no transplants from deceased donors. We terminate the procedure.

Based on this procedure, we state the following theorem:

Theorem 4 (Waiting times under regular and incentivized exchange) *Suppose Assumptions 2 and 3 hold. Consider the ABO-i deceased-donor allocation and incentivized-exchange policies with a given incentivized-exchange participation-rate vector* $\rho = (\rho_{X-Y})_{Y \triangleright X, Y \neq X}$ *(which can possibly be zero). Consider a blood type X. Then the following statements hold:*

1. *Blood-type X patients, who are in overdemanded-type or self-demanded-type pairs and who have either incompatible donors or are eligible and willing to participate in incentivized exchange, participate in exchange immediately upon their arrival to the exchange pool.*
2. *Suppose the patients in pairs of underdemanded types X-Y_1,...,X-$Y_{\ell(X)}$ receive both deceased-donor and exchange transplants, while the patients in pairs of underdemanded types X-$Y_{\ell(X)+1}$,...,X-$Y_{k(X)}$ receive only exchange transplants for some $\ell(X) \in \{0,\ldots,k(X)\}$. Then*
   - *nonprioritized & unpaired type X patients and the patients of type X-Y_1,...,X-$Y_{\ell(X)}$ pairs wait for a deceased-donor (or exchange) transplant for the duration*

$$t_X^{\mathbf{i},dec} = F^{-1}\left(1 - \frac{\delta_X - \phi^l \mathbf{c}_X + \pi^{\mathbf{i}}_{Y_1-X} + \ldots + \pi^{\mathbf{i}}_{Y_{\ell(X)}-X}}{\pi^{\mathbf{i}}_X + \pi^{\mathbf{i}}_{X-Y_1} + \ldots + \pi^{\mathbf{i}}_{X-Y_{\ell(X)}}}\right), \text{ and,} \tag{6}$$

   - *for all $\ell \in \{\ell(X)+1,\ldots,k(X)\}$, type X-$Y_\ell$ pairs are exclusively matched through exchange and wait for an exchange transplant for the duration*

$$t^{\mathbf{i}}_{X-Y_\ell} = F^{-1}\left(1 - \frac{\pi^{\mathbf{i}}_{Y_\ell-X}}{\pi^{\mathbf{i}}_{X-Y_\ell}}\right) \leq t_X^{\mathbf{i},dec}. \tag{7}$$

The average waiting time to a transplant for all type X patients is

$$t_X^{\mathbf{i},ave} = \frac{\left[\delta_X - \phi^l \mathbf{c}_X + \sum_{\ell=1}^{\ell(X)} \pi^{\mathbf{i}}_{Y_\ell-X}\right] t_X^{\mathbf{i},dec} + \sum_{\ell=\ell(X)+1}^{k(X)} \left[\pi^{\mathbf{i}}_{Y_\ell-X}\, t^{\mathbf{i}}_{X-Y_\ell}\right]}{\delta_X + \mathbf{l}_X + \mathbf{e}_X + \mathbf{i}_X} \tag{8}$$

Proof. The proof follows from the procedure discussed before the statement of the theorem. Since type X patients with compatible paired donors and type X patients with incompatible but blood-type-compatible donors have 0 waiting time, Equation 8 is established. ■

When $\rho = 0$, we will refer to $t_X^{\mathbf{i},dec}$ as $t_X^{\mathbf{e},dec}$ and $t_X^{\mathbf{i},ave}$ as $t_X^{\mathbf{e},ave}$.

A.3 Waiting Times under Balanced Incentivized Exchange

On the other hand, under balanced incentivized exchange, compatible pairs of types X-Y $\in$ {A-O, B-O, AB-O, AB-A, AB-B}, who participate in exchange and whose patients reenter later due to transplant failures, are prioritized in the type Y deceased-donor queue, instead of the type X deceased-donor queue.

In this case, in some equations given in the previous section, the term $\mathbf{c}_X$ should be replaced with a new term. While Equation 3 does not change, in Equations 4 (the new entity defined is referred to as $r_X^{\mathbf{b}}$) and 5 (the new entity defined is referred to as $r^{\mathbf{b}}_{X,X-Y_1,\ldots,X-Y_\ell}$), $\mathbf{c}_X$ should be replaced with $\mathbf{c}_X^{\mathbf{b}}$, which is defined as

$$\mathbf{c}_X^{\mathbf{b}} = \sum_{Y\,:\,X \triangleright Y \,\&\, X \neq Y} \rho_{Y-X}(1-\theta) p_X \lambda_Y \pi_Y. \tag{9}$$

Similarly, in Equations 6 (the new entity defined is referred to as $t_X^{\mathbf{b},dec}$) and 8 (the new entity defined is referred to as $t_X^{\mathbf{b},ave}$) in Theorem 4, $\mathbf{c}_X$ should be replaced with $\mathbf{c}_X^{\mathbf{b}}$. As such, the pooling procedure above and Theorem 4 continue to hold.

While contrasting $\mathbf{c}_X^{\mathbf{b}}$ with $\mathbf{c}_X$ defined in Equation 2, observe that $\mathbf{c}_X^{\mathbf{b}}$ is the flow of all paired type Y patients with compatible type X donors who participate in incentivized exchange and help

the patient of a type X-Y pair to receive a transplant. If these patients reenter the queue, they are prioritized in front of nonprioritized & unpaired type X patients, and this explains the $-\phi^l \mathbf{c}^{\mathbf{b}}_X$ term in the numerator of the new versions of Equations 4, 5, 6, and 8. On the other hand, the term $\mathbf{c}_X$ remains in Equation 3 as is because all incentivized reentering X patients are prioritized at some deceased-donor queue and drop out of competition for the remaining type X deceased donors. Thus, the $\pi^{\mathbf{i}}_X$ expression in Equation 3 remains as the flow of nonprioritized & unpaired type X patients. We have

$$\begin{aligned}
\mathbf{c}^{\mathbf{b}}_O &= \rho_{A-O}(1-\theta)p_O\lambda_A\pi_A + \rho_{B-O}(1-\theta)p_O\lambda_B\pi_B + \rho_{AB-O}(1-\theta)p_O\lambda_{AB}\pi_{AB},\\
\mathbf{c}^{\mathbf{b}}_A &= \rho_{AB-A}(1-\theta)p_A\lambda_{AB}\pi_{AB},\\
\mathbf{c}^{\mathbf{b}}_B &= \rho_{AB-B}(1-\theta)p_B\lambda_{AB}\pi_{AB}, \text{ and}\\
\mathbf{c}^{\mathbf{b}}_{AB} &= 0.
\end{aligned}$$

A.4 Equity Implications of Transplant Regimes on Access of Patients with Living Donors

Proof of Theorem 2. Let $p_A > p_B$. Suppose that $\lambda_X = \lambda$ for any blood type X, and $\frac{\pi_X}{\pi_Y} = \frac{p_X}{p_Y}$ for any two blood types X,Y. Also assume that the fraction of pairs taking the incentivized exchange is uniform at a fixed $\rho < 1$ for any eligible type.

1. We consider $\mathbf{l}_X$, the overall flows of pairs with X blood type participating in direct living-donation technology for each blood type X:

$$\begin{aligned}
\frac{\mathbf{l}_O}{\pi_O} &= \frac{(1-\theta)p_O\lambda\pi_O}{\pi_O} = (1-\theta)p_O\lambda,\\
\frac{\mathbf{l}_A}{\pi_A} &= \frac{(1-\theta)(p_O+p_A)\lambda\pi_A}{\pi_A} = (1-\theta)(p_O+p_A)\lambda,\\
\frac{\mathbf{l}_B}{\pi_B} &= \frac{(1-\theta)(p_O+p_B)\lambda\pi_B}{\pi_B} = (1-\theta)(p_O+p_B)\lambda, \text{ and}\\
\frac{\mathbf{l}_{AB}}{\pi_{AB}} &= \frac{(1-\theta)\lambda\pi_{AB}}{\pi_{AB}} = (1-\theta)\lambda.
\end{aligned}$$

Thus,

$$\frac{\mathbf{l}_O}{\pi_O} < \frac{\mathbf{l}_A}{\pi_A}, \frac{\mathbf{l}_B}{\pi_B} < \frac{\mathbf{l}_{AB}}{\pi_{AB}}.$$

Moreover, since $p_B < p_A$, we have $\frac{\mathbf{l}_B}{\pi_B} < \frac{\mathbf{l}_A}{\pi_A}$.

2. We consider $\mathbf{e}_X$, the overall flows of pairs that have type X patients and participate in regular

exchange, for each X:

$$\begin{aligned}
\frac{\mathbf{e}_O}{\pi_O} =& \frac{\theta p_O \lambda(\pi_O + \pi_A + \pi_B + \pi_{AB})}{\pi_O} = (\theta p_O + \theta p_A + \theta p_B + \theta p_{AB})\,\lambda = \theta\lambda, \\
\frac{\mathbf{e}_A}{\pi_A} =& \frac{\theta p_O \lambda \pi_A + \theta p_A \lambda \pi_A + p_B \lambda \pi_A + \theta p_A \lambda \pi_{AB}}{\pi_A} = (\theta p_O + \theta p_A + p_B + \theta p_{AB})\,\lambda, \\
\frac{\mathbf{e}_B}{\pi_B} =& \frac{\theta p_O \lambda \pi_B + p_B \lambda \pi_A + \theta p_B \lambda \pi_B + \theta p_B \lambda \pi_{AB}}{\pi_B} = (\theta p_O + p_A + \theta p_B + \theta p_{AB})\lambda, \text{ and} \\
\frac{\mathbf{e}_{AB}}{\pi_{AB}} =& \frac{\theta p_O \lambda \pi_{AB} + \theta p_A \lambda \pi_{AB} + \theta p_B \lambda \pi_{AB} + \theta p_{AB} \lambda \pi_{AB}}{\pi_{AB}} = (\theta p_O + \theta p_A + \theta p_B + \theta p_{AB})\,\lambda = \theta\lambda,
\end{aligned}$$

where the second equality in each line (except the last) follows from the assumption that $\frac{p_X}{\pi_X}$ is constant among all X. Since $\theta < 1$ and $p_A, p_B > 0$, we have

$$\frac{\mathbf{e}_O}{\pi_O} = \frac{\mathbf{e}_{AB}}{\pi_{AB}} < \frac{\mathbf{e}_A}{\pi_A}, \frac{\mathbf{e}_B}{\pi_B}.$$

With the additional assumption $p_A > p_B$, we obtain $\frac{\mathbf{e}_A}{\pi_A} < \frac{\mathbf{e}_B}{\pi_B}$.

We consider each $\mathbf{l}_X + \mathbf{e}_X$, the flow of direct living-donor and exchange transplants in total. We have

$$\begin{aligned}
\frac{\mathbf{l}_O + \mathbf{e}_O}{\pi_O} =& (1-\theta)p_O\lambda + \theta(p_O + p_A + p_B + p_{AB})\lambda = (p_O + \theta p_A + \theta p_B + \theta p_{AB})\lambda, \\
\frac{\mathbf{l}_A + \mathbf{e}_A}{\pi_A} =& (1-\theta)(p_O + p_A)\lambda + (\theta p_O + \theta p_A + p_B + \theta p_{AB})\lambda = (p_O + p_A + p_B + \theta p_{AB})\lambda, \\
\frac{\mathbf{l}_B + \mathbf{e}_B}{\pi_B} =& (1-\theta)(p_O + p_B)\lambda + (\theta p_O + p_A + \theta p_B + \theta p_{AB})\lambda = (p_O + p_A + p_B + \theta p_{AB})\lambda, \text{ and} \\
\frac{\mathbf{l}_{AB} + \mathbf{e}_{AB}}{\pi_{AB}} =& (1-\theta)\lambda + \theta\lambda = \lambda.
\end{aligned}$$

Since $\theta < 1$ and $p_A, p_B, p_{AB} > 0$,

$$\frac{\mathbf{l}_O + \mathbf{e}_O}{\pi_O} < \frac{\mathbf{l}_A + \mathbf{e}_A}{\pi_A} = \frac{\mathbf{l}_B + \mathbf{e}_B}{\pi_B} < \frac{\mathbf{l}_{AB} + \mathbf{e}_{AB}}{\pi_{AB}} = \lambda.$$

3. Next we consider $\mathbf{i}_X$, the overall flow of pairs with type X patients benefitting from incentivized exchange for each blood type X:

$$\begin{aligned}
\frac{\mathbf{i}_O}{\pi_O} =& \frac{\rho(1-\theta)p_O\lambda\pi_A + \rho(1-\theta)p_O\lambda\pi_B + \rho(1-\theta)p_O\lambda\pi_{AB}}{\pi_O} = \rho(1-\theta)(p_A + p_B + p_{AB})\lambda, \\
\frac{\mathbf{i}_A}{\pi_A} =& \frac{\rho(1-\theta)p_A\lambda\pi_{AB}}{\pi_A} = \rho(1-\theta)p_{AB}\lambda, \\
\frac{\mathbf{i}_B}{\pi_B} =& \frac{\rho(1-\theta)p_B\lambda\pi_{AB}}{\pi_B} = \rho(1-\theta)p_{AB}\lambda, \text{ and} \\
\frac{\mathbf{i}_{AB}}{\pi_{AB}} =& 0,
\end{aligned}$$

where the second equality in each line (except the last) follows from the assumption that $\frac{p_X}{\pi_X}$ is constant among all X. Since $\rho > 0$, $\theta < 1$, and $p_A, p_B, p_{AB} > 0$,

$$0 = \frac{\mathbf{i}_{AB}}{\pi_{AB}} < \frac{\mathbf{i}_A}{\pi_A} = \frac{\mathbf{i}_B}{\pi_B} < \frac{\mathbf{i}_O}{\pi_O}.$$

We consider each $\mathbf{l}_X + \mathbf{e}_X + \mathbf{i}_X$, i.e., direct living-donor, regular-exchange, and incentivized-

exchange transplants in total. We have

$$\begin{aligned}
\frac{\mathbf{l}_O + \mathbf{e}_O + \mathbf{i}_O}{\pi_O} =& p_O\lambda + [\theta + \rho(1-\theta)]\,(p_A + p_B + p_{AB})\lambda, \\
\frac{\mathbf{l}_A + \mathbf{e}_A + \mathbf{i}_A}{\pi_A} =& (p_O + p_A + p_B)\lambda + [\theta + \rho(1-\theta)]\,p_{AB}\lambda, \\
\frac{\mathbf{l}_B + \mathbf{e}_B + \mathbf{i}_B}{\pi_B} =& (p_O + p_A + p_B)\lambda + [\theta + \rho(1-\theta)]\,p_{AB}\lambda, \text{ and} \\
\frac{\mathbf{l}_{AB} + \mathbf{e}_{AB} + \mathbf{i}_{AB}}{\pi_{AB}} =& \lambda.
\end{aligned}$$

Since $\rho, p_{AB}, \lambda > 0$,

$$\frac{\mathbf{l}_O + \mathbf{e}_O + \mathbf{i}_O}{\pi_O} < \frac{\mathbf{l}_A + \mathbf{e}_A + \mathbf{i}_A}{\pi_A} = \frac{\mathbf{l}_B + \mathbf{e}_B + \mathbf{i}_B}{\pi_B} < \frac{\mathbf{l}_{AB} + \mathbf{e}_{AB} + \mathbf{i}_{AB}}{\pi_{AB}} = \lambda,$$

and they are all equal if and only if $\rho = 1$, because $\theta < 1$. ■

A.5 Consequences of Different Transplant Regimes on Waiting Times

In this subsection, we state a lemma that formalizes the marginal effects of living-donor exchange policies on the waiting times of the following 29 groups of patients under some reasonable assumptions. These 29 groups are nonprioritized & unpaired patients of each blood type (4 groups), compatible pairs of overdemanded and self-demanded types (5 groups for overdemanded types and 4 groups for self-demanded types; recall that the overdemanded type A-B pairs are never compatible), incompatible pairs of overdemanded and self-demanded types (6 groups for overdemanded types and 4 groups for self-demanded types), and pairs of underdemanded types (6 groups).

In addition to Assumptions 2 and 3, we also assume that the tissue-type incompatibility probability θ and the reentry rate of living-donation recipients ϕ^l are sufficiently small. Formally, "for a vector of **sufficiently small** parameters x, some claim holds" means that "there exists some vector $\overline{x} \gg 0$ (i.e., all entries of the vector are larger than 0) such that for all x, $0 \leq x \leq \overline{x}$, that claim holds." These assumptions guarantee that all underdemanded-type pairs, except possibly type B-A, are pooled with their respective nonprioritized & unpaired patients for deceased donation under the regular-exchange technology. Furthermore, we also assume that the difference between flows of pair types B-A and A-B is sufficiently small. This guarantees that B-A pairs only participate in exchange and are never pooled for deceased donation in all of exchange regimes we consider. This lemma will be used to prove Theorem 3.

Lemma 2 (Consequences of regular and incentivized exchange) *Suppose Assumptions 2 and 3 hold for a given incentivized-exchange participation-fraction profile $\rho^* = (\rho^*_{X-Y})_{Y \triangleright X \& Y \neq X}$. Suppose also that the reentry fraction of living-donation recipients, ϕ^l, the flow difference between types B-A and A-B, $p_A\lambda_B\pi_B - p_B\lambda_A\pi_A$, and the tissue-type incompatibility probability, θ, are sufficiently small. Then the following results hold:*

- *With respect to living-donation technology, regular-exchange technology causes steady-state waiting times of all nonprioritized & unpaired patient groups and all incompatible pair groups to decrease. In particular, in addition to compatible pair groups, all incompatible overdemanded*

and self-demanded pair groups no longer wait for a transplant and receive exchange transplants immediately upon entry.

- *With respect to regular-exchange technology, incentivized-exchange technology causes the waiting times of*
  - *all overdemanded- and self-demanded-type pair groups to stay at zero,*
  - *all underdemanded-type pair groups except type B-A pairs to decrease,*
  - *type B-A pairs not to change,*
  - *nonprioritized & unpaired type O, type A, and type B patient groups to decrease, and*
  - *nonprioritized & unpaired type AB patient group to increase.*
- *With respect to regular-exchange technology, balanced-incentivized-exchange technology causes the waiting times of*
  - *all overdemanded- and self-demanded-type pair groups to stay at zero,*
  - *all underdemanded-type pair groups except type B-A pairs to decrease,*
  - *type B-A pairs not to change,*
  - *nonprioritized & unpaired type O, type A, and type B patient groups to decrease, and*
  - *nonprioritized & unpaired type AB patient group not to change.*

Proof of Lemma 2. Suppose we fix an incentivized-exchange participation-fraction vector $\rho^* = (\rho^*_{X-Y})_{Y \triangleright X \& Y \neq X}$ such that Assumptions 2 and 3 hold. Then under any of the exchange policies (i.e., regular with $\rho = 0$, incentivized with $\rho = \rho^*$, or balanced incentivized with $\rho = \rho^*$), the flow of underdemanded-type X-Y pairs and their reciprocal-type Y-X pairs (from Equation 1) satisfy:

$$\pi^{\mathbf{i}}_{X-Y} = p_Y \lambda_X \pi_X \geq \pi^{\mathbf{i}}_{Y-X} = \begin{cases} [\theta + \rho_{Y-X}(1-\theta)] p_X \lambda_Y \pi_Y & \text{if Y-X} \neq \text{A-B} \\ p_B \lambda_A \pi_A & \text{otherwise} \end{cases}.$$

As we have established before, in the optimal, ABO-i exchange regime for each of the three exchange policies, none of the pairs of incompatible overdemanded and self-demanded types wait for a transplant, as they immediately receive transplant through exchange.

In the rest of the proof, we focus on the other patient groups: underdemanded-type pairs and nonprioritized & unpaired patients.

Suppose also that the tissue-type incompatibility probability, θ, and the flow difference between types B-A and A-B, $p_A \lambda_B \pi_B - p_B \lambda_A \pi_A$, are sufficiently small.

We prove the following claim first:

Claim 1. Under regular exchange, patients of all underdemanded-types pairs except that of B-A are pooled for deceased donation with the nonprioritized & unpaired patients of the same blood type, while patients of type B-A pairs are never pooled under any exchange regime.

Proof of Claim 1. For a blood type X $\in$ {O, A, B} (note that type AB patients are not in

any underdemanded-type pairs), under regular exchange we have $\mathbf{c}_X\big|_{\rho=0} = 0$. We also have

$$r_X\big|_{\rho=0} = \frac{\delta_X}{\pi_X^{\mathbf{i}}\big|_{\rho=0}} > r_{X-Y}\big|_{\rho=0} = \frac{\theta p_X \lambda_Y \pi_Y}{p_Y \lambda_X \pi_X} \tag{10}$$

for any underdemanded type X-Y $\neq$ B-A, where the inequality follows for sufficiently small θ.

Recall that $k(A) = 1$ and $k(O) = 3$ are the numbers of underdemanded pair types with type A and type O patients, respectively.

Thus, pairs of the only underdemanded type with A patient, A-AB, are pooled with nonprioritized & unpaired type A patients under regular exchange by Equation 10.

We order underdemanded pair types with patient blood type O according to the ascending order of their r ratios as O-Y_1, O-Y_2, and O-Y_3. Then, for $\ell = 1, 2$,

$$r_{O,O-Y_1,\ldots,O-Y_\ell}\big|_{\rho=0} = \frac{\delta_O + \sum_{m=1}^{\ell} \theta p_O \lambda_{Y_m} \pi_{Y_m}}{\pi_O^{\mathbf{i}}\big|_{\rho=0} + \sum_{m=1}^{\ell} p_{Y_m} \lambda_O \pi_O} > \frac{\theta p_O \lambda_{Y_{\ell+1}} \pi_{Y_{\ell+1}}}{p_{Y_{\ell+1}} \lambda_O \pi_O} = r_{O-Y_{\ell+1}}\big|_{\rho=0} \tag{11}$$

because of the assumption that θ is sufficiently small.

Thus, under regular exchange, underdemanded-type pairs with type O patients are pooled for deceased donation with nonprioritized & unpaired type O patients.

On the other hand, for the underdemanded pair type B-A, we have

$$r_{B-A} = \frac{p_B \lambda_A \pi_A}{p_A \lambda_B \pi_B} > r_B \tag{12}$$

for any ρ, when the difference $p_A \lambda_B \pi_B - p_B \lambda_A \pi_A$ is sufficiently small. Thus, pairs of type B-A are never pooled with nonprioritized & unpaired type B patients under incentivized exchange for any ρ. The same holds for balanced incentivized exchange, when we replace, in Equation 12, r_B with $r_B^{\mathbf{b}}$. Thus, pairs of type B-A are never pooled with nonprioritized & unpaired type B patients under balanced incentivized exchange for any ρ, either.

Equation 10 with X = B and Equation 12 imply that pairs of type B-AB are pooled with nonprioritized & unpaired type B patients under regular exchange. □

We also assume that the fraction of living-donation recipients reentering the deceased-donor queue, ϕ^l, is also sufficiently small in the rest of the proof.

Transition from Living-Donor Transplantation to Regular Exchange:

Consider a blood type X. The flow of pairs that benefit from direct living donation is given by $\mathbf{l}_X = \sum_{Y:Y \triangleright X} (1-\theta) p_Y \lambda_X \pi_X$. The flow of pairs that benefit from regular exchange satisfies $\mathbf{e}_X = \sum_{Y:Y \triangleright X} \theta p_Y \lambda_X \pi_X + \sum_{Y:X \triangleright Y, Y \neq X} \theta p_X \lambda_Y \pi_Y + \mathbf{1}_{\left\{X \in \{A,B\}\right\}} p_B \lambda_A \pi_A > 0$.[17] This is also the flow of patients that fall out of competition from the type X deceased-donor queue with respect to the living-donor transplantation technology.

We consider the ratio of the available deceased-donor flow to the flow of patients who cannot receive direct living donation, which we refer to as $r_X^{\mathbf{l}}$, and $r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}|_{\rho=0}}\big|_{\rho=0}$ when pairs of un-

[17] The indicator function $\mathbf{1}_{\{Z\}}$gets value 1 if the event Z is true and value 0 if the event Z is false.

derdemanded types X-Y_1, . . . ,X-$Y_{\ell(X)|_{\rho=0}}$ are pooled for deceased donation under regular exchange. We have

$$r_X^{\mathbf{l}} = \frac{\delta_X}{\pi_X - \underbrace{\sum_{Y:Y\triangleright X}(1-\theta)p_Y\lambda_X\pi_X}_{=\mathbf{l}_X} + \phi^d\delta_X + \phi^l\mathbf{l}_X} \quad \text{and} \tag{13}$$

$$r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\big|_{\rho=0} = \frac{\delta_X + \sum_{m=1}^{\ell(X)|_{\rho=0}}(\theta p_X\lambda_{Y_m}\pi_{Y_m})}{\underbrace{\pi_X - \lambda_X\pi_X + \phi^d\delta_X + \phi^l\mathbf{l}_X + \phi^l\mathbf{e}_X}_{=\pi_X^{\mathbf{i}}\big|_{\rho=0}} + \sum_{m=1}^{\ell(X)|_{\rho=0}}(p_{Y_m}\lambda_X\pi_X)}. \tag{14}$$

Claim 2. The waiting time decreases under regular exchange with respect to living donation for unpaired patients and underdemanded-type pairs.

Proof of Claim 2. For all X, we have from Equations 13 and 14 that, when $\phi^l = 0$

$$r_X^{\mathbf{l}}\big|_{\phi^l=0} = \frac{\delta_X}{\pi_X - (1-\theta)\sum_{Y:Y\triangleright X}p_Y\lambda_X\pi_X + \phi^d\delta_X} \text{ and}$$

$$r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\big|_{\rho=0,\phi^l=0} = \frac{\delta_X + \sum_{m=1}^{\ell(X)\big|_{\rho=0,\phi^l=0}}(\theta p_X\lambda_{Y_m}\pi_{Y_m})}{\pi_X - (1-\sum_{m=1}^{\ell(X)\big|_{\rho=0,\phi^l=0}}p_{Y_m})\lambda_X\pi_X + \phi^d\delta_X}.$$

Since B-A pairs are not pooled for deceased donation by Claim 1, we have B-A $\neq$ X-Y_m for any X and m. Thus, for each Y_m, X $\triangleright$ Y_m and $Y_m \neq$ X. Thus, we obtain $1-\sum_{m=1}^{\ell(X)\big|_{\rho=0,\phi^l=0}}p_{Y_m} \geq 1-\sum_{Y:X\triangleright Y,Y\neq X}p_Y$. We also have $1-\sum_{Y:X\triangleright Y,Y\neq X}p_Y = \sum_{Y:Y\triangleright X}p_Y > (1-\theta)\sum_{Y:Y\triangleright X}p_Y$, as $\theta > 0$. Thus,

$$r_X^{\mathbf{l}}\big|_{\phi^l=0} < r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\big|_{\rho=0,\phi^l=0}.$$

By the continuity of the r ratios in ϕ^l, for sufficiently small ϕ^l we have $r_X^{\mathbf{l}} < r_X\big|_{\rho=0}$, implying that

$$t_X^{\mathbf{l},dec} = F^{-1}\left(1-r_X^{\mathbf{l}}\right) > F^{-1}\left(1-r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\big|_{\rho=0}\right) = t_X^{\mathbf{e},dec}. \tag{15}$$

Since by Claim 1 pairs of underdemanded types except B-A are pooled with deceased donation under regular exchange, their waiting times also decrease. Moreover, the waiting time of type B-A pairs decreases even more, as it is not pooled for deceased donation by Claim 1.[18] □

Transition from Regular Exchange to Incentivized Exchange:

Consider a blood type X $\in$ {A, B, O}. Suppose ρ^* is the participation profile for incentivized exchange. The flow of pairs who benefit from incentivized exchange with any ρ in addition to regular exchange satisfies

$$\mathbf{i}_X = \sum_{Y:X\triangleright Y,Y\neq X}\rho_{Y-X}(1-\theta)p_X\lambda_Y\pi_Y,$$

[18] Claim 2 is actually stronger for blood type AB. We established in the main text that the claim is true for blood type AB regardless of the magnitudes of θ and ϕ^l.

while the flow of prioritized reentrants satisfies

$$\phi^l \mathbf{c}_X = \phi^l \left(\sum_{Y: Y \triangleright X, Y \neq X} \rho_{X-Y}(1-\theta) p_Y \lambda_X \pi_X \right).$$

As a result, for some $\ell(X) \in \{0, \ldots, k(X)\}$, pairs of underdemanded types X-Y_1,...,X-$Y_{\ell(X)}$ are pooled for deceased donation at ρ, and thus, we have

$$r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}} = \frac{\delta_X - \phi^l \mathbf{c}_X + \sum_{m=1}^{\ell(X)} \left([\theta + \rho_{Y_m - X}(1-\theta)] p_X \lambda_{Y_m} \pi_{Y_m} \right)}{\underbrace{\pi_X - \lambda_X \pi_X + \phi^d \delta_X + \phi^l \mathbf{l}_X + \phi^l \mathbf{e}_X + \phi^l \mathbf{i}_X}_{=\pi_X^{\mathbf{i}}} + \sum_{m=1}^{\ell(X)} p_{Y_m} \lambda_X \pi_X}. \tag{16}$$

Claim 3. The waiting time decreases under incentivized exchange with respect to regular exchange for nonprioritized & unpaired X patients and all underdemanded-type pairs with X patients — except type B-A.

Proof of Claim 3. We will show that all ratios r_{X-Y_m} for all $m = 1, ..., k(X)$, such that X-$Y_m \neq$ B-A, and ratio $r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}$ increase from $\rho = 0$ to $\rho = \rho^*$, and thus, the related waiting time decreases. We have $\ell(X) = \ell(X)\big|_{\rho=0}$ (i.e., the number of pooled types at $\rho = 0$) for sufficiently small ρ profiles, since r ratios are continuous around $\rho = 0$ and there are no sudden jumps in pooling by Claim 1. Thus, for sufficiently small ρ, when $\phi^l = 0$

$$r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\big|_{\phi^l=0} = \frac{\delta_X + \sum_{m=1}^{\ell(X)|_{\phi^l=0}} \left([\theta + \rho_{Y_m - X}(1-\theta)] p_X \lambda_{Y_m} \pi_{Y_m} \right)}{\pi_X - \lambda_X \pi_X + \phi^d \delta_X + \sum_{m=1}^{\ell(X)|_{\phi^l=0}} p_{Y_m} \lambda_X \pi_X}$$

and

$$r_{X-Y_m} = \frac{[\theta + \rho_{Y_m - X}(1-\theta)] p_X \lambda_{Y_m} \pi_{Y_m}}{p_{Y_m} \lambda_X \pi_X}$$

are increasing in ρ. Suppose that we increase each ρ_{W-Z} from 0 to ρ^*_{W-Z} in uniform speed ρ^*_{W-Z} throughout so that ρ reaches ρ^* at time $t = 1$. We can compare the rates of change in both entities along this line as the inner product of their gradient vector and speed vector:

$$\begin{aligned}\left(\nabla_\rho r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}} \right) \cdot \rho^*\big|_{\phi^l=0} &= \frac{\sum_{m=1}^{\ell(X)|_{\phi^l=0}} \rho^*_{Y_m - X}(1-\theta) p_X \lambda_{Y_m} \pi_{Y_m}}{\pi_X - \lambda_X \pi_X + \phi^d \delta_X + \sum_{m=1}^{\ell(X)|_{\phi^l=0}} p_{Y_m} \lambda_X \pi_X} \\ &< \frac{\rho^*_{Y_m - X}(1-\theta) p_X \lambda_{Y_m} \pi_{Y_m}}{p_{Y_m} \lambda_X \pi_X} = (\nabla_\rho r_{X-Y_m}) \cdot \rho^*,\end{aligned}$$

for $m = \ell(X)|_{\phi^l=0}$, i.e., the r ratio for the pooled nonprioritized & unpaired patients and underdemanded-type pairs changes slower than the largest of the r ratios of the underdemanded types that are pooled when $\phi^l = 0$. Thus, as ρ increases to ρ^*, either ρ reaches ρ^* without $\ell(X)|_{\phi^l=0}$ changing or there will be a profile ρ^1 such that $0 \ll \rho^1 < \rho^*$, at which $\ell(X)|_{\phi^l=0}$ decreases to $\ell(X)\big|_{\rho=0,\phi^l=0} - 1$ so that pairs of the underdemanded type with the highest r ratio are no longer pooled with the rest. Similarly, the resulting new r value relevant for the pool of nonprioritized & unpaired patients and remaining underdemanded-type pairs will be increasing in ρ until ρ reaches a new cutoff $\rho^2 \leq \rho^*$. At this new cutoff $\ell(X)\big|_{\rho=\rho^2,\phi^l=0} = \ell(X)\big|_{\rho=0,\phi^l=0} - 2$, and so on, so forth. Possibly, no underdemanded pairs may remain pooled at sufficiently high ρ, implying that $\ell(X)|_{\phi^l=0} = 0$,

and thus, $\left(\nabla_\rho r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\right)\cdot\rho^*\big|_{\phi^l=0}=0$. Except after this last iteration, all r ratios strictly increase at each iteration until $t=1$ at different speeds when $\phi^l=0$.

In the end, for sufficiently small ϕ^l, by the continuity of the r ratios (and their gradients) in ϕ^l and by the fact that all underdemanded-type pairs were pooled initially at $\rho=0$, all gradients are strictly positive at least for small ρ. Thus, we obtain that the r ratios strictly increase from $\rho=0$ to $\rho=\rho^*$. As the waiting time is decreasing in its relevant r ratio for each patient group, for all underdemanded types with X patient blood type — except type B-A — and nonprioritized & unpaired X patients, waiting times strictly decrease with respect to their levels at $\rho=0$. □

On the other hand, in the main text in Section 4 we showed that the waiting time of nonprioritized & unpaired type AB patients strictly increases from $\rho=0$ to $\rho=\rho^*$ regardless of ϕ^l and θ. For sufficiently small type B-A and type A-B flow difference, since pairs of type B-A are not pooled with nonprioritized & unpaired B patients regardless of ρ^* by Claim 1, their waiting time remains unaffected for any ρ, including $\rho=0$ and $\rho=\rho^*$.

Transition from Regular Exchange to Balanced Incentivized Exchange:

The flow of underdemanded-type pairs who benefit from balanced incentivized exchange with participation fraction profile ρ^* satisfies

$$\mathbf{b}_X=\mathbf{i}_X=\sum_{Y:X\triangleright Y,Y\neq X}\rho_{Y-X}(1-\theta)p_X\lambda_Y\pi_Y,$$

while the flow of prioritized reentrants in front of the type X deceased-donor queue satisfies

$$\phi^l\mathbf{c}^{\mathbf{b}}_X|_{\rho=\rho^*}=\phi^l\sum_{Y:X\triangleright Y,Y\neq X}{\rho^*}_{Y-X}(1-\theta)p_X\lambda_Y\pi_Y.$$

Then Claim 3's proof follows line by line after replacing $\mathbf{c}^{\mathbf{b}}_X\big|_{\rho=\rho^*}$ with $\mathbf{c}_X\big|_{\rho=\rho^*}$ in Equation 16 for X $\in$ {O, A, B}.

On the other hand, for type X = AB, since $\mathbf{c}^{\mathbf{b}}_{AB}\big|_{\rho=\rho^*}=0$, $\mathbf{b}_{AB}\big|_{\rho=\rho^*}=0$, and there are no underdemanded types, the relevant ratio in Equation 16 reduces to

$$r^{\mathbf{b}}_{AB}\big|_{\rho=\rho^*}=\frac{\delta_{AB}}{\underbrace{\pi_{AB}-\lambda_{AB}\pi_{AB}+\phi^d\delta_{AB}+\phi^l\mathbf{l}_{AB}+\phi^l\mathbf{e}_{AB}}_{=\pi^{\mathbf{b}}_{AB}\big|_{\rho=\rho^*}=\pi^{\mathbf{i}}_{AB}\big|_{\rho=0}}}=r_{AB}\big|_{\rho=0},$$

which is also the relevant ratio for regular exchange. Therefore, $t^{\mathbf{b},dec}_{AB}\big|_{\rho=\rho^*}=F^{-1}\left(1-r^{\mathbf{b}}_{AB}\big|_{\rho=\rho^*}\right)=F^{-1}\left(1-r_{AB}\big|_{\rho=0}\right)=t^{\mathbf{e},dec}_{AB}$, completing the proof of the lemma. ■

A.6 Equity Consequences of Different Transplant Regimes on Deceased-Donor Queues

Proof of Theorem 3. Suppose Assumptions 2 and 3 hold. Let $p_A>p_B$. Suppose that $\lambda_X=\lambda$ for any blood type X, and $\frac{\delta_X}{\delta_Y}=\frac{\pi_X}{\pi_Y}=\frac{p_X}{p_Y}$ for any two blood types X,Y. Also assume that the fraction of pairs taking the incentivized-exchange option is uniform at a fixed $\rho<1$ for any eligible pair type.

1. ***With deceased-donor transplantation only***, the waiting time at each deceased-donor queue is $t_X^{\mathbf{d},dec} = F^{-1}\left(1 - \frac{\delta_X}{\pi_X + \phi^d \delta_X}\right) = F^{-1}\left(1 - \frac{\frac{\delta_X}{\pi_X}}{1+\phi^d \frac{\delta_X}{\pi_X}}\right)$ for any blood type X. Since $\frac{\delta_X}{\pi_X} = \frac{\delta_Y}{\pi_Y}$ for any two blood types X and Y, we have $t_X^{\mathbf{d},dec} = t_Y^{\mathbf{d},dec}$.
2. ***Introduction of living-donor transplantation*** reduces the waiting time at each deceased-donor queue X, since $t_X^{\mathbf{l},dec} = F^{-1}\left(1 - \frac{\frac{\delta_X}{\pi_X}}{1+\phi^d \frac{\delta_X}{\pi_X} - (1-\phi^l)p_X^l \lambda}\right) < F^{-1}\left(1 - \frac{\frac{\delta_X}{\pi_X}}{1+\phi^d \frac{\delta_X}{\pi_X}}\right) = t_X^{\mathbf{d},dec}$. Since the probability of being compatible with the paired donor conditional on having a living donor satisfies for each blood type

$$p_O^l = (1-\theta)p_O, \qquad p_B^l = (1-\theta)(p_O + p_B),$$
$$p_A^l = (1-\theta)(p_O + p_A), \qquad p_{AB}^l = (1-\theta)(p_O + p_A + p_B + p_{AB}) = (1-\theta),$$

and $p_A > p_B$, we have $p_O^l < p_B^l < p_A^l < p_{AB}^l$. Thus, as $t_X^{\mathbf{l},dec}$ is decreasing in p_X^l and $\frac{\delta_X}{\pi_X}$ is constant among blood types, we have

$$t_{AB}^{\mathbf{l},dec} < t_A^{\mathbf{l},dec} < t_B^{\mathbf{l},dec} < t_O^{\mathbf{l},dec}.$$

Moreover, Part 1 implies that

$$\left(t_{AB}^{\mathbf{d},dec} - t_{AB}^{\mathbf{l},dec}\right) > \left(t_A^{\mathbf{d},dec} - t_A^{\mathbf{l},dec}\right) > \left(t_B^{\mathbf{d},dec} - t_B^{\mathbf{l},dec}\right) > \left(t_O^{\mathbf{d},dec} - t_O^{\mathbf{l},dec}\right).$$

Further assume that θ and ϕ^l are sufficiently small in the rest of the proof. We also have the flow difference between type B-A and type A-B as $p_B\lambda\pi_A - p_A\lambda\pi_B = 0$ since $\frac{p_A}{p_B} = \frac{\pi_A}{\pi_B}$. Thus, hypothesis of Lemma 2 holds.

3. ***Introduction of regular exchange***, in addition to direct living-donor transplantation and deceased-donor transplantation, causes the deceased-donor waiting times for all blood types to decrease by Lemma 2. By Claim 1 in the proof of the same lemma, pairs of all underdemanded types except B-A are pooled for deceased donation with unpaired patients of their patients' blood types. By Equation 14 and $\frac{\pi_X}{\pi_Y} = \frac{p_X}{p_Y}$ for any two blood types X and Y, we obtain

$$\begin{aligned} r_{O,O-A,O-B,O-AB}\big|_{\rho=0} &= \frac{\delta_O + (\theta p_O \lambda \pi_A + \theta p_O \lambda \pi_B + \theta p_O \lambda \pi_{AB})}{\pi_O - \lambda\pi_O + \phi^d\delta_O + \phi^l(\mathbf{l}_O + \mathbf{e}_O) + (p_A\lambda\pi_O + p_B\lambda\pi_O + p_{AB}\lambda\pi_O)} \\ &= \frac{\frac{\delta_O}{\pi_O} + (\theta p_A\lambda + \theta p_B\lambda + \theta p_{AB}\lambda)}{1 - \lambda + \phi^d\frac{\delta_O}{\pi_O} + \phi^l\left(\frac{\mathbf{l}_O + \mathbf{e}_O}{\pi_O}\right) + (p_A\lambda + p_B\lambda + p_{AB}\lambda)}, \end{aligned}$$

$$r_{A,A-AB}\big|_{\rho=0} = \frac{\delta_A + (\theta p_A\lambda\pi_{AB})}{\pi_A - \lambda\pi_A + \phi^d\delta_A + \phi^l(\mathbf{l}_A + \mathbf{e}_A) + (p_{AB}\lambda\pi_A)} = \frac{\frac{\delta_A}{\pi_A} + (\theta p_{AB}\lambda)}{1 - \lambda + \phi^d\frac{\delta_A}{\pi_A} + \phi^l\left(\frac{\mathbf{l}_A + \mathbf{e}_A}{\pi_A}\right) + (p_{AB}\lambda)},$$

$$r_{B,B-AB}\big|_{\rho=0} = \frac{\delta_B + (\theta p_B\lambda\pi_{AB})}{\pi_B - \lambda\pi_B + \phi^d\delta_B + \phi^l(\mathbf{l}_B + \mathbf{e}_B) + (p_{AB}\lambda\pi_B)} = \frac{\frac{\delta_B}{\pi_B} + (\theta p_{AB}\lambda)}{1 - \lambda + \phi^d\frac{\delta_B}{\pi_B} + \phi^l\left(\frac{\mathbf{l}_B + \mathbf{e}_B}{\pi_B}\right) + (p_{AB}\lambda)},$$

$$r_{AB} = \frac{\delta_{AB}}{\pi_B - \lambda\pi_{AB} + \phi^d\delta_{AB} + \phi^l(\mathbf{l}_{AB} + \mathbf{e}_{AB})} = \frac{\frac{\delta_{AB}}{\pi_{AB}}}{1 - \lambda + \phi^d\frac{\delta_{AB}}{\pi_{AB}} + \phi^l\left(\frac{\mathbf{l}_{AB} + \mathbf{e}_{AB}}{\pi_{AB}}\right)}.$$

Since $\frac{\pi_X}{\pi_Y} = \frac{\delta_X}{\delta_Y}$ for any two blood types X and Y, we have by Theorem 2, $\frac{\mathbf{l}_A + \mathbf{e}_A}{\pi_A} = \frac{\mathbf{l}_B + \mathbf{e}_B}{\pi_B}$, and

thus, $r_{A,A-AB}\big|_{\rho=0} = r_{B,B-AB}\big|_{\rho=0}$ implying that

$$t_A^{\mathbf{e},dec} = F^{-1}\left(1 - r_{A,A-AB}\big|_{\rho=0}\right) = F^{-1}\left(1 - r_{B,B-AB}\big|_{\rho=0}\right) = t_B^{\mathbf{e},dec}.$$

Suppose $\phi^l = 0$. Then,

$$r_{O,O-A,O-B,O-AB}\big|_{\rho=0,\phi^l=0} = \frac{\frac{\delta_O}{\pi_O} + (\theta p_A\lambda + \theta p_B\lambda + \theta p_{AB}\lambda)}{1-\lambda+\phi^d\frac{\delta_O}{\pi_O} + (p_A\lambda + p_B\lambda + p_{AB}\lambda)},$$

$$r_{A,A-AB}\big|_{\rho=0,\phi^l=0} = \frac{\frac{\delta_A}{\pi_A} + (\theta p_{AB}\lambda)}{1-\lambda+\phi^d\frac{\delta_A}{\pi_A} + (p_{AB}\lambda)}, \text{ and}$$

$$r_{AB}\big|_{\phi^l=0} = \frac{\frac{\delta_{AB}}{\pi_{AB}}}{1-\lambda+\phi^d\frac{\delta_{AB}}{\pi_{AB}}}.$$

Since for sufficiently small θ, $\frac{\delta_X/\pi_X}{1-\lambda+\phi^d\delta_X/\pi_X} > \theta$ for any X, we have that $\frac{\delta_X/\pi_X+\theta q(X)}{1-\lambda+\phi^d\delta_X/\pi_X+q(X)}$ is decreasing in $q(X)$ for any $q(X) \geq 0$. Thus, we can rank the above entities as $r_{O,O-A,O-B,O-AB}\big|_{\rho=0,\phi^l=0} < r_{A,A-AB}\big|_{\rho=0,\phi^l=0} < r_{AB}\big|_{\phi^l=0}$. By the continuity of these r ratios in ϕ^l, for sufficiently small ϕ^l we still have $r_{O,O-A,O-B,O-AB}\big|_{\rho=0} < r_{A,A-AB}\big|_{\rho=0} < r_{AB}$. As the generic waiting time $t = F^{-1}(1-r)$ is decreasing in r, we can rank the waiting times for deceased donation in the queue under regular exchange as

$$t_{AB}^{\mathbf{e},dec} < t_A^{\mathbf{e},dec} = t_B^{\mathbf{e},dec} < t_O^{\mathbf{e},dec},$$

and thus, by Part 1,

$$\left(t_{AB}^{\mathbf{d},dec} - t_{AB}^{\mathbf{e},dec}\right) > \left(t_A^{\mathbf{d},dec} - t_A^{\mathbf{e},dec}\right) = \left(t_B^{\mathbf{d},dec} - t_B^{\mathbf{e},dec}\right) > \left(t_O^{\mathbf{d},dec} - t_O^{\mathbf{e},dec}\right).$$

4.(a) ***Introduction of incentivized exchange*** in addition to regular exchange, direct living-donor transplantation, and deceased-donor transplantation causes the waiting time for a deceased-donor transplant to decrease for all blood types except AB, for which it increases by Lemma 2, with respect to regular exchange. Since $\frac{p_X}{p_Y} = \frac{\pi_X}{\pi_Y}$ for any two blood types X and Y, the relevant r ratios for waiting times in the deceased-donor queue satisfy for each $k = 0, ..., k(X)$, such that X-$Y_k \neq$ B-A,

$$r_{X,X-Y_1,...,X-Y_k} = \frac{\frac{\delta_X}{\pi_X} - \phi^l\frac{\mathbf{c}_X}{\pi_X} + \sum_{m=1}^{k}\left([\theta+\rho(1-\theta)]p_{Y_m}\lambda\right)}{1-\lambda+\phi^d\frac{\delta_X}{\pi_X} + \phi^l\frac{\mathbf{l}_X+\mathbf{e}_X+\mathbf{i}_X}{\pi_X} + \sum_{m=1}^{k}p_{Y_m}\lambda},$$

where $\frac{\mathbf{l}_O+\mathbf{e}_O+\mathbf{i}_O}{\pi_O} < \frac{\mathbf{l}_A+\mathbf{e}_A+\mathbf{i}_A}{\pi_A} = \frac{\mathbf{l}_B+\mathbf{e}_B+\mathbf{i}_B}{\pi_{AB}} < \frac{\mathbf{l}_{AB}+\mathbf{e}_{AB}+\mathbf{i}_B}{\pi_B} = \lambda$ by Theorem 2, and

$$\frac{\mathbf{c}_O}{\pi_O} = 0 \;\; < \frac{\mathbf{c}_A}{\pi_A} = \rho(1-\theta)p_O\lambda \;\; = \frac{\mathbf{c}_B}{\pi_B} = \rho(1-\theta)p_O\lambda \;\; < \frac{\mathbf{c}_{AB}}{\pi_{AB}} = \rho(1-\theta)(p_O+p_A+p_B)\lambda.$$

Moreover, we have that for all underdemanded types X-Y except type B-A, the r ratio

$$r_{X-Y} = \frac{[\theta+\rho(1-\theta)]p_X\lambda\pi_Y}{p_Y\lambda\pi_X} = \theta + \rho(1-\theta) \tag{17}$$

is uniform. Define $\hat{r}_X := r_{X,X-Y_1,...,X-Y_{\ell(X)}}$. Thus, type A-AB pairs will be pooled with nonprioritized & unpaired type A patients if and only if type B-AB pairs will be pooled with nonprioritized & unpaired type B patients, implying $\hat{r}_A = \hat{r}_B$, and hence,

$$t_A^{\mathbf{i},dec} = F^{-1}\left(1-\hat{r}_A\right) = F^{-1}\left(1-\hat{r}_B\right) = t_B^{\mathbf{i},dec}.$$

(b) ***Introduction of balanced incentivized exchange***, in addition to regular exchange, direct living-donor transplantation, and deceased-donor transplantation, causes the waiting time for a deceased-donor transplant to decrease for all blood types except AB, for which it remains constant by Lemma 2, with respect to regular exchange.

We next inspect the magnitude of these changes.

Note that $\mathbf{b}_X = \mathbf{i}_X$ for each blood type X, i.e., the same flow of pairs benefit from both balanced-incentivized-exchange and incentivized-exchange regimes. Given a blood type X, the relevant r ratio for waiting times in the deceased-donor queue, which we will refer to as r_X^* for short, satisfies

$$r_X^* := r^{\mathbf{b}}_{X,X-Y_1,\ldots,X-Y_{\ell^{\mathbf{b}}(X)}} = \frac{\frac{\delta_X}{\pi_X} - \phi^l \frac{\mathbf{c}^{\mathbf{b}}{}_X}{\pi_X} + \sum_{m=1}^{\ell^{\mathbf{b}}(X)} \left([\theta + \rho(1-\theta)] p_{Y_m} \lambda\right)}{1 - \lambda + \phi^d \frac{\delta_X}{\pi_X} + \phi^l \frac{\mathbf{l}_X + \mathbf{e}_X + \mathbf{i}_X}{\pi_X} + \sum_{m=1}^{\ell^{\mathbf{b}}(X)} p_{Y_m} \lambda}.$$

We prove several claims:

Claim 1. Deceased-donation waiting times in the A and B deceased-donor queues are the same under incentivized balanced exchange, i.e., $t_A^{\mathbf{b},dec} = t_B^{\mathbf{b},dec}$.

Proof of Claim 1. We have $\frac{\mathbf{l}_O + \mathbf{e}_O + \mathbf{i}_O}{\pi_O} < \frac{\mathbf{l}_A + \mathbf{e}_A + \mathbf{i}_A}{\pi_A} = \frac{\mathbf{l}_B + \mathbf{e}_B + \mathbf{i}_B}{\pi_B} < \frac{\mathbf{l}_{AB} + \mathbf{e}_{AB} + \mathbf{i}_B}{\pi_{AB}} = \lambda$ by Theorem 2 and $\frac{\delta_X}{\delta_Y} = \frac{\pi_X}{\pi_Y}$ for any two blood types X and Y. Moreover, $\frac{\mathbf{c}^{\mathbf{b}}{}_X}{\pi_X}$ satisfies

$$\frac{\mathbf{c}_O^{\mathbf{b}}}{\pi_O} = \rho(1-\theta)(p_A + p_B + p_{AB})\lambda \qquad > \frac{\mathbf{c}_A^{\mathbf{b}}}{\pi_A} = \rho(1-\theta)p_{AB}\lambda = \frac{\mathbf{c}_B^{\mathbf{b}}}{\pi_B} \qquad > \frac{\mathbf{c}_{AB}^{\mathbf{b}}}{\pi_{AB}} = 0.$$

Moreover, Equation 17 holds in this case as well. Thus, type A-AB pairs will be pooled with nonprioritized & unpaired type A patients if and only if type B-AB pairs will be pooled with nonprioritized & unpaired type B patients, Hence, $r_A^* = r_B^*$, implying that

$$t_A^{\mathbf{b},dec} = F^{-1}\left(1 - r_A^*\right) = F^{-1}\left(1 - r_B^*\right) = t_B^{\mathbf{b},dec}.$$

□

Thus, without loss of generality, we will focus on blood type A among A and B in the rest of the proof, i.e., everything we prove for blood type A will also hold for blood type B.

Claim 2. Suppose $\phi^l = 0$. Let $\overline{\rho} = \frac{\left(r_{AB}^*|_{\phi^l=0}\right) - \theta}{1-\theta}$. Then

- if $\rho < \overline{\rho}$, all underdemanded types will be pooled with their respective patient blood types and $r_O^*\big|_{\phi^l=0} < r_A^*\big|_{\phi^l=0} < r_{AB}^*\big|_{\phi^l=0}$ such that each of these r ratios is increasing in ρ, and
- if $\rho \geq \overline{\rho}$, no underdemanded type will be pooled and $r_O^*\big|_{\phi^l=0} = r_A^*\big|_{\phi^l=0} = r_{AB}^*\big|_{\phi^l=0}$ is constant in ρ.

Proof of Claim 2. We have the relevant r ratios for the waiting time for deceased donation

for each blood type O, A, and AB as follows for any ϕ^l:

$$r_O^* = \frac{\frac{\delta_O}{\pi_O} - \phi^l\rho(1-\theta)(p_A+p_B+p_{AB})\lambda + \sum_{m=1}^{\ell^{\mathbf{b}}(O)}\Big(\big[\theta+\rho(1-\theta)\big]p_{Y_m}\lambda\Big)}{1-\lambda+\phi^d\frac{\delta_O}{\pi_O}+\phi^l\Big(p_O\lambda+\big[\theta+\rho(1-\theta)\big](p_A+p_B+p_{AB})\lambda\Big)+\sum_{m=1}^{\ell^{\mathbf{b}}(O)}p_{Y_m}\lambda},$$

$$r_A^* = \frac{\frac{\delta_A}{\pi_A} - \phi^l\rho(1-\theta)p_{AB}\lambda + \sum_{m=1}^{\ell^{\mathbf{b}}(A)}\Big(\big[\theta+\rho(1-\theta)\big]p_{Y_m}\lambda\Big)}{1-\lambda+\phi^d\frac{\delta_A}{\pi_A}+\phi^l\Big((p_O+p_A+p_B)\lambda+\big[\theta+\rho(1-\theta)\big]p_{AB}\lambda\Big)+\sum_{m=1}^{\ell^{\mathbf{b}}(A)}p_{Y_m}\lambda}, \text{ and} \tag{18}$$

$$r_{AB}^* = \frac{\frac{\delta_{AB}}{\pi_{AB}}}{1-\lambda+\phi^d\frac{\delta_{AB}}{\pi_{AB}}+\phi^l\lambda}.$$

Suppose $\phi^l = 0$. First, observe that for all underdemanded types X-Y other than B-A Equation 17 holds, and thus, we have $r_{X-Y} = [\theta+\rho(1-\theta)]$. Moreover, the relevant r ratio regarding nonprioritized & unpaired patient groups, when all deceased donors are served to them and pairs of no underdemanded type are pooled with them, satisfies (for D and P defined as below):

$$\begin{aligned} r_A^{\mathbf{b}}\big|_{\phi^l=0} &= \frac{\overbrace{\frac{\delta_A}{\pi_A}}^{=D}}{\underbrace{1-\lambda+\phi^d\frac{\delta_A}{\pi_A}}_{=P}} = \frac{D}{P} \\ &= r_O^{\mathbf{b}}\big|_{\phi^l=0} = r_{AB}^*\big|_{\phi^l=0} \end{aligned} \tag{19}$$

because $\frac{\pi_X}{\pi_Y} = \frac{\delta_X}{\delta_Y}$ for all X and Y.
Thus,

$$r_A^{\mathbf{b}}\big|_{\phi^l=0} = \frac{D}{P} > r_{A-AB} = \theta+\rho(1-\theta) \qquad \Longleftrightarrow \qquad r_O^{\mathbf{b}}\big|_{\phi^l=0} = \frac{D}{P} > r_{O-X} = \theta+\rho(1-\theta)$$

for any X $\in$ {A, B, AB}. This means A-AB pairs will be pooled for deceased donation with A patients if and only if all underdemanded-type pairs with O patients will be pooled for deceased donation with O patients, in turn, each of these statements is equivalent to

$$\frac{D}{P} > \theta+\rho(1-\theta). \tag{20}$$

Define

$$\overline{\rho} = \frac{\frac{D}{P}-\theta}{1-\theta}. \tag{21}$$

Two cases are possible for ρ.

- Case 1. $\rho < \overline{\rho}$: Then Inequality 20 holds, and pairs of all underdemanded types except B-A are pooled with their respective patients for deceased donation. Moreover, $r_X^*|_{\phi^l=0}$ can be written as

$$r_X^*\big|_{\phi^l=0} = \frac{D+[\theta+\rho(1-\theta)]q(X)}{P+q(X)}, \tag{22}$$

  where $q(X) = \sum_{m=1}^{\ell^{\mathbf{b}}(X)} p_{Y_m}\lambda$. Since $\rho < \overline{\rho}$, we have $r_X^{\mathbf{b}}|_{\phi^l=0} = \frac{D}{P} > \theta+\rho(1-\theta) = r_{X-Y}$ for

all underdemanded pair types X-Y except B-A (where equalities follow by Equations 19 and 17). Thus, $r_X^*\big|_{\phi^l=0}$ is decreasing in $q(X)$. Since

$$q(AB) = 0 < q(A) = p_{AB}\lambda < q(O) = (p_A + p_B + p_{AB})\lambda,$$

we obtain

$$r_O^*\big|_{\phi^l=0} < r_A^*\big|_{\phi^l=0} < r_{AB}^*\big|_{\phi^l=0}.$$

Finally, each of these is increasing in ρ by Equation 22.

- Case 2. $\rho \geq \overline{\rho}$: Then Inequality 20 will not hold, and there will be no pooling of underdemanded type pairs. Thus, $r_X^*|_{\phi^l=0} = r_X^{\mathbf{b}}|_{\phi^l=0} = \frac{D}{P}$ is constant in ρ. □

Claim 3. Under balanced-incentivized-exchange technology, the deceased-donation waiting times are ranked as follows: $t_{AB}^{\mathbf{b},dec} < t_A^{\mathbf{b},dec} = t_B^{\mathbf{b},dec} < t_O^{\mathbf{b},dec}$.

Proof of Claim 3. Claim 1 shows $t_A^{\mathbf{b},dec} = t_B^{\mathbf{b},dec}$. First, we will show that $r_O^* < r_A^* < r_{AB}^*$. For X ∈ {O, A}, we have two cases for ρ (for $\overline{\rho}$ defined as in Equation 21 and D, P defined as in Equation 19):

- Case 1. $\rho < \overline{\rho}$: $r_X^*\big|_{\phi^l=0} = \frac{D+[\theta+\rho(1-\theta)]q(X)}{P+q(X)}$ by Equation 22. For sufficiently small ϕ^l, we have the ranking $r_O^* < r_A^* < r_{AB}^*$ by Claim 2 and the continuity of the r ratios in ϕ^l.
- Case 2. $\rho \geq \overline{\rho}$: $r_X^*\big|_{\phi^l=0} = \frac{D}{P}$ is constant by Claim 2. Thus, we need to consider $\frac{\partial r_X^*}{\partial \phi^l}$ evaluated at $\phi^l = 0$ to find the ranking of r_X^* for sufficiently small ϕ^l using Equation system 18 and the fact that no underdemanded-type pair is pooled in this case (from Claim 2). We have:

$$\frac{\partial r_O^*}{\partial \phi^l}\Big|_{\phi^l=0} = \frac{-\rho(1-\theta)(p_A+p_B+p_{AB})\lambda P - \Big(p_O + \big[\theta+\rho(1-\theta)\big]\big(p_A+p_B+p_{AB}\big)\Big)\lambda D}{P^2} < 0,$$

$$\frac{\partial r_A^*}{\partial \phi^l}\Big|_{\phi^l=0} = \frac{-\rho(1-\theta)p_{AB}\lambda P - \Big(\big(p_O+p_A+p_B\big) + \big[\theta+\rho(1-\theta)\big]p_{AB}\Big)\lambda D}{P^2} < 0, \text{ and}$$

$$\frac{\partial r_{AB}^*}{\partial \phi^l}\Big|_{\phi^l=0} = \frac{-\lambda D}{P^2} < 0.$$

Moreover,

$$\frac{\partial r_O^*}{\partial \phi^l}\Big|_{\phi^l=0} - \frac{\partial r_A^*}{\partial \phi^l}\Big|_{\phi^l=0} = -\frac{(1-\theta)(p_A+p_B)\lambda}{P}\left(\rho\Big(1+\frac{D}{P}\Big) - \frac{D}{P}\right),$$

and since $1 - p_{AB} = p_O + p_A + p_B$,

$$\frac{\partial r_A^*}{\partial \phi^l}\Big|_{\phi^l=0} - \frac{\partial r_{AB}^*}{\partial \phi^l}\Big|_{\phi^l=0} = -\frac{(1-\theta)p_{AB}\lambda}{P}\left(\rho\Big(1+\frac{D}{P}\Big) - \frac{D}{P}\right).$$

Suppose $\theta = 0$. Since $\rho > \overline{\rho}|_{\theta=0} = \frac{D}{P}$, we have $\rho(1+\frac{D}{P}) - \frac{D}{P} > \frac{D}{P}(1+\frac{D}{P}) - \frac{D}{P} = \big(\frac{D}{P}\big)^2 > 0$. Thus,

$$\frac{\partial r_O^*}{\partial \phi^l}\Big|_{\phi^l=0,\theta=0} < \frac{\partial r_A^*}{\partial \phi^l}\Big|_{\phi^l=0,\theta=0} < \frac{\partial r_{AB}^*}{\partial \phi^l}\Big|_{\phi^l=0,\theta=0} < 0.$$

Moreover,

$$\frac{\partial r_O^*}{\partial \theta}\Big|_{\phi^l=0,\theta=0} = \frac{\partial r_A^*}{\partial \theta}\Big|_{\phi^l=0,\theta=0} = \frac{\partial r_{AB}^*}{\partial \theta} = 0.$$

Hence, by the Taylor series expansion of the r ratios in (ϕ^l, θ) around $(0,0)$, for sufficiently small (ϕ^l, θ) we have $r_O^* < r_A^* < r_{AB}^*$. Since $t_X^{\mathbf{b},dec} = F^{-1}(1 - r_X^*)$ and by Claim 1 $t_A^{\mathbf{b},dec} = t_B^{\mathbf{b},dec}$, we obtain

$$t_O^{\mathbf{b},dec} < t_A^{\mathbf{b},dec} = t_B^{\mathbf{b},dec} < t_{AB}^{\mathbf{b},dec}.$$

□

Finally, we complete the proof of Part 4(b) and the theorem as follows: Since $t_{min}^{\mathbf{b},dec} = t_{AB}^{\mathbf{b},dec} = t_{AB}^{\mathbf{e},dec} = t_{min}^{\mathbf{e},dec}$ and $t_{max}^{\mathbf{b},dec} = t_O^{\mathbf{b},dec} < t_O^{\mathbf{e},dec} = t_{max}^{\mathbf{e},dec}$, we obtain

$$\left(t_{max}^{\mathbf{b},dec} - t_{min}^{\mathbf{b},dec}\right) < \left(t_{max}^{\mathbf{e},dec} - t_{min}^{\mathbf{e},dec}\right).$$

■

Appendix B Construction of Calibration Parameters for Numerical Predictions

In this appendix, we explain how the calibration parameters, reported in Table 1 in Section 5 and used to generate the numerical model predictions, are constructed.

In Table 6, we report the blood-type distribution for different ethnicities and fractions of these ethnicities in the US population. Using these, we calculate an overall US blood-type distribution (the last row of this table). We use this as the blood-type distribution of living donors, (p_X), in our model.

US Blood Type and Ethnicity Distribution Data					
Ethnicities	**Blood Types**				**Pop. %**
	O	**A**	**B**	**AB**	
African American	0.490	0.270	0.200	0.040	12.4%
Asian American	0.400	0.280	0.270	0.050	3.3%
Native American	0.790	0.160	0.040	0.010	0.8%
White American	0.450	0.400	0.110	0.040	83.4%
US population	0.456	0.378	0.126	0.040	

Table 6: The US blood type and ethnicity distribution. Retrieved from `http://bloodbook.com` on 03/05/2018. The blood-type distribution for the overall US population is constructed using the ethnicity distribution and could be slightly different from the general distributions reported in other sources.

In Table 7, we report the OPTN data for deceased-donor queue additions and deceased- and living-donor transplants for the year of 2009. We measure time in 1 year units and calculate the flows using the annual numbers reported in this data. First, we observe that on average $\frac{2\times 7248}{10442} = 1.4407$ kidneys are harvested from each deceased donor, since a total of 7248 deceased donors arrive while 10442 deceased-donor transplants are conducted. The deceased-donor flows, (δ_X), are constructed by multiplying each entry in the second to last row of the table with 1.4407. The row above, deceased-donation recipients, is used as the de-facto deceased-donor flows, (δ'_X), in the numerical calculations.

New patient arrival flows, (π_X), are calculated as follows: We know the annual additions to the deceased-donor queue (the first row of the table). However, some patients receive living-donor

The US OPTN Kidney Data					
	O	**A**	**B**	**AB**	**Overall**
Total Additions to the Queue	16323	11090	4919	1325	33657
Living-Donation Recipients not on Queue	432	406	127	35	1000
Reentrants	2062	1513	580	198	4353
Direct Living-Donation Recipients	2750	2326	786	236	5098
Exchange Participants	128	96	58	8	290
Deceased-Donation Recipients	4726	3815	1347	554	10442
Deceased Donors	3458	2722	850	218	7248
Average CPRA					4.73%

Table 7: Arrivals to and transplants from the kidney deceased-donor queue for 2009 entrants. Year 2009 is used because this is the latest year for which five-year dialysis survival rates are available as of May, 2017. Data is obtained from the OPTN using the "national data" option from `http://optn.transplant.hrsa.gov` 05/12/2017.

transplants without even registering in the queue (the second row of the table). We add these two numbers and subtract the number of reentrants (the third row of the table) from them to find π_X for each blood type X.

Reentry fractions, ϕ^l and ϕ^d, are assumed to be the same, as the OPTN national data do not distinguish reentrants based on their previous transplantation type. We divide the total number of reentrants (the last cell of the third row of the table) by the sum of direct living-donation recipients, exchange participants, and deceased-donation recipients (the sum of the last cells of the fourth, fifth, and sixth lines of the table).

The tissue-type incompatibility probability, θ, is taken as the average calculated panel reactive antibody (CPRA), 0.0473, for the 2009 entrants (see Table 9 in Appendix C for its calculations using the OPTN data). CPRA measures the percentage of the population with which the patient is tissue-type incompatible.

The calculation of paired-donor fractions, (λ_X), requires the knowledge of the total number of patients who arrive with paired donors. However, this information is not available since only the realized living-donor transplants are recorded in this database. Most of these transplants are direct transplants, i.e., those from the compatible paired donor of a patient. A small percentage of those are from exchanges.[19] In the fourth row of Table 7, we report the numbers of direct living donations conducted (i.e., each entry is $\mathbf{l}_X$ in our model). Assuming patients and living donors are paired initially as in our model, we calculate the probability of having a compatible donor conditional on being paired with a living donor. These probabilities are calculated as follows using the living-donor blood-type distribution, (p_X), reported in the last row of Table 6:

$$p^l_O = (1-\theta)p_O = 0.4344, \qquad p^l_B = (1-\theta)(p_O + p_B) = 0.5545,$$
$$p^l_A = (1-\theta)(p_O + p_A) = 0.7974, \text{ and} \qquad p^l_{AB} = (1-\theta) = 0.9527.$$

Then, we calculate $\lambda_X = \frac{\mathbf{l}_X}{p^l_X \pi_X}$ for each blood type X.

The incentivized-exchange participation fraction for a compatible pair type X-Y with Y $\triangleright$ X and

[19] In 2009, exchange transplants were still rare in the US, consisting of about 4.54 percent of all living-donor transplants, while this percentage is more than 12 percent as of 2016.

Y $\neq$ X, ρ_{X-Y}, is our free calibration variable. We assume that this fraction is uniform for each type, and we denote it as ρ. We consider three treatments with $\rho = 25\%$, 50%, and 100%.

US On-dialysis Survival Rates					
Time:	3 mo.	1 yr.	2 yr.	3 yr.	5 yr.
On dialysis	0.9215	0.7824	0.6648	0.5694	0.4230

Table 8: For the survival rate, $1 - F(t)$, we use the on-dialysis survival probability of US kidney disease patients for the 2009 entrant cohort. These 2009 estimates for dialysis patients are obtained from the weighted average of hemodialysis and peritoneal dialysis survival rates reported in the 2016 Annual Report of the National Kidney Organization, Volume 2, Table 6.3 of Chapter 6 (retrieved from `https://www.usrds.org/2016/download/v2_c06_Mortality_16.pdf` on 05/12/2017). The weights used in the weighted average calculations are the average 2009–2014 percentages of patients on hemodialysis versus peritoneal dialysis, reported in the same report's Volume 2, Figure 1.2 of Chapter 1 (retrieved from `https://www.usrds.org/2016/download/v2_c01_IncPrev_16.pdf` on 05/12/2017).

In Table 8, survival rates over time, $1 - F(t)$, are reported. These are obtained from the US Renal Data System (USRDS) data for dialysis patients. We fit an exponential duration curve (for t measured in years) as

$$\hat{F}(t) = 1 - \hat{a}e^{\hat{b}t}$$

and obtain the following estimates using non-linear least squares regression (through the `fit` command in MATLAB): $\hat{a} = 0.9427$ with the 95 percent confidence interval $(0.8945, 0.9909)$, and $\hat{b} = -0.1667$ with the 95 percent confidence interval $(-0.1922, -0.1411)$.

Appendix C Simulations

In addition to the numerical model predictions in Section 5, we also conduct simulations emulating the discrete paired and unpaired patient and deceased-donor arrival processes in real life. Our goal in conducting these simulations is to assess the welfare and equity consequences of our policy proposal, incentivized exchange, more accurately. Moreover, the simulations give us a chance to assess the validity of our continuum model in conducting policy analysis.

C.1 Simulation Methodology

In the simulations, we distribute deceased-donor kidneys according to the de-facto allocation policy on a FIFO basis to a compatible patient. If no compatible patient is found in the queue, the kidney immediately perishes. Our exchange policy is ABO-i two-way exchange, the optimal mechanism for our model: Each arriving eligible type X-Y pair is immediately matched with a mutually compatible type Y-X pair selected on a FIFO basis from the ones waiting in the queue. If such a mutually compatible match does not exist and the type X-Y pair is a compatible pair that wanted to take the incentivized exchange option, then it does not wait in the queue and its donor directly donates to its patient. In this case, the patient shall not be prioritized if he reenters the queue with a transplant failure. On the other hand, if the type X-Y pair is incompatible, then it joins both the deceased-donor and exchange queues.

We assume patients are heterogenous in their tissue-type incompatibility probabilities. We use the entrant CPRA distribution reported in Table 9 to generate the tissue-type incompatibility probability θ_i for each patient i. The mean of this distribution gives us the value of θ we use in the numerical model predictions, 0.0473 (or 4.73 percentage points in the CPRA reporting metric), in Section 5. This data gives the fraction of entrants in five different CPRA intervals. We assume that all patients uniformly and randomly take CPRA values in their assigned CPRA intervals. For example, the US OPTN data reports that 1.73 percent of all entrants have CPRA points between 0 percent and 20 percent (the second column of this table). We first assume that a simulated patient i is assigned to this group with probability 0.0173. Then his exact tissue-type incompatibility probability θ_i is determined uniformly randomly from the interval $(0, 0.2)$.

The US OPTN Data for CPRA Distribution for Entrants					
	CPRA intervals (in % points)				
	0	**(0,20)**	**[20,80)**	**[80,98)**	**[98,100)**
Fraction of Entrants	91.57%	1.73%	3.85%	1.88%	0.98%

Table 9: Data is obtained from the OPTN using the "national data" option for the year 2009 from `http://optn.transplant.hrsa.gov` on 03/28/2018.

Our simulations use a scaled-down version of the calibrated inflow rates for new patients and deceased donors. The US consists of 13 transplant regions of various sizes. Deceased-donor kidneys are first offered to patients within their arrival regions. If a suitable match cannot be found in the region, then they are offered nationally. Our simulation roughly maps to one region that comprises one tenth of the size of the US and reflects the same patient and donor characteristics as the overall US population does. Thus, we obtain deceased-donor and new-patient arrival flows by dividing the population flows δ'_X and π_X reported in Table 1 by 10. For other parameters of the simulation, (p_X), (λ_X), ϕ^l, ϕ^d, and $1 - F(t)$, we use the same parameters reported in this table.

In each iteration, we simulate the evolution of the kidney-allocation process in such a region for 55 years. Each year is divided into finite periods so that in each period either only one new patient, reentrant, or deceased donor arrives. Thus, the number of periods in each year equals the sum of the total flow of new patients, $\sum_X \pi_X/10$, the total flow of deceased-donor kidneys, $\sum_X \delta'_X/10$, and the total number of reentrants. The number of reentrants in the next year is calculated as the minimum of (a) the reentry fraction ϕ multiplied by the number of total transplants in the previous year and (b) the total number of patients who received a transplant since year 1. The numbers of patients waiting in the queue, periods per year, and reentrants per year stabilize after a number of years passes. For example, beginning from the end of year 31 until the end of last year, year 55, the number of patients waiting in the deceased-donor queue at the year's end has a standard deviation less than 0.5 percent of its mean. This is approximately the steady state we are seeking. We report the averages of the last year (year 55) for the numbers and percentages of patients matched. For calculating average waiting times, we need for all of the patients of a cohort to exit the queue, either with a transplant or without a transplant, i.e., because of death. We observe that in each of the treatments and simulations no patient waits more than 25 years in the queue. We report the

average waiting time of patients who arrive in year 31. Thus, when the simulation ends at year 55, all patients have exited the queue in year 31 entry cohort with or without a transplant. We run a total of 100 simulations and report their averages and standard errors.

The new-patient, deceased-donor, and reentrant generation processes are as follows: Each new patient is generated independently and randomly with the underlying blood-type, tissue-type incompatibility probability, and living-donor pairing probability distributions. We also randomly determine his survival time while waiting for a transplant so that the population probability of remaining alive after waiting for t years is $1 - F(t)$. Once a patient is deemed paired, his living donor's blood type is also independently randomly generated in a similar fashion. We determine whether they are compatible using their blood types and the patient's tissue-type incompatibility probability with a random donor. For a deceased-donor kidney, we only generate its blood type according to the distribution dictated by (δ'_X). A reentrant is determined according to the reentry probability among the transplanted patients since year 1 with uniform distribution. For a reentrant, we use his original tissue-type incompatibility probability and blood type. We assume that he is now unpaired. We also calculate his new survival time using the same overall survival probability function $1 - F(t)$.

We consider nine treatments in our simulations. The first six treatments are deceased-donor transplantation, living-donor transplantation, regular exchange, and balanced-incentivized exchange for uniform participation rates $\rho = 25\%$, 50%, 100%. These were also used in our numerical model predictions. We also consider three additional balanced-incentivized exchange treatments in which compatible type X-X pairs are also incentivized. In our continuum model, this incentivization scheme does not have additional welfare benefits, as all incompatible type X-X pairs are matched with each other in regular exchange as soon as they arrive. On the other hand, in our simulations, as pair arrivals are discrete and patients are heterogenous in their tissue-type compatibility probabilities, there could be potential welfare gains from the participation of compatible type X-X pairs in exchange with incompatible X-X pairs already in the queue.

C.2 Simulation Results

The simulation results are very similar to the numerical model predictions for the de-facto deceased-donor allocation policy. The numbers and percentages of transplanted patients are given in Table 10. When compared with Table 2 in Section 5, the corresponding percentages are slightly higher for the simulations. The new treatment, balanced incentivized exchange with compatible X-X pairs, fares only slightly better than the balanced-incentivized treatment. The average waiting times for a nonprioritized deceased-donor transplant and any type of transplant are displayed in Tables 11 and 12, respectively. Similarly, they are slightly shorter than their counterparts predicted by our model in Section 5. The balanced-incentivized treatment with compatible X-X pairs makes almost no difference in waiting times for nonprioritized deceased-donor transplants, but slightly shortens overall waiting times.

Simulation Results: Patients Receiving Transplant											
		O		**A**		**B**		**AB**		**Overall**	
Treatments		***Living-Donor Transplants***									
Living-donor transplantation ($\mathbf{l}_X$)		273.92 (16.00)	*18.68% (0.99%)*	232.82 (14.15)	*23.22% (1.32%)*	76.81 (9.11)	*17.33% (1.81%)*	22.98 (4.93)	*19.82% (3.99%)*	606.53 (23.02)	*20.03% (0.75%)*
Regular exchange ($\mathbf{e}_X + \mathbf{l}_X$)		290.70 (17.20)	*19.83% (1.08%)*	278.95 (16.39)	*27.82% (1.50%)*	117.18 (11.73)	*26.46% (2.39%)*	23.90 (5.06)	*20.61% (4.09%)*	710.73 (28.64)	*23.48% (0.93%)*
Balanced Incentivized ($\mathbf{e}_X + \mathbf{l}_X + \mathbf{b}_X$)	$\rho = 25\%$	340.98 (17.12)	*23.26% (1.11%)*	280.78 (16.99)	*28.00% (1.57%)*	117.98 (12.00)	*26.64% (2.47%)*	23.93 (5.02)	*20.64% (4.06%)*	763.67 (31.59)	*25.22% (1.03%)*
	$\rho = 50\%$	390.20 (18.56)	*26.62% (1.24%)*	282.94 (16.77)	*28.22% (1.55%)*	118.51 (12.01)	*26.76% (2.47%)*	23.88 (5.10)	*20.59% (4.10%)*	815.53 (33.30)	*26.94% (1.08%)*
	$\rho = 100\%$	487.91 (20.71)	*33.29% (1.42%)*	287.33 (17.08)	*28.66% (1.56%)*	119.96 (11.93)	*27.09% (2.46%)*	23.88 (5.07)	*20.59% (4.06%)*	919.08 (38.03)	*30.36% (1.23%)*
Balanced Incentivized also with comp. type X-X pairs ($\mathbf{e}_X + \mathbf{l}_X + \mathbf{b}_X$)	$\rho = 25\%$	344.85 (17.01)	*23.53% (1.11%)*	282.39 (16.93)	*28.16% (1.55%)*	118.31 (11.93)	*26.71% (2.46%)*	23.96 (5.08)	*20.66% (4.10%)*	769.51 (31.58)	*25.42% (1.03%)*
	$\rho = 50\%$	395.16 (18.17)	*26.96% (1.21%)*	285.05 (16.93)	*28.43% (1.54%)*	119.04 (11.91)	*26.88% (2.43%)*	23.91 (5.10)	*20.61% (4.10%)*	823.16 (32.52)	*27.19% (1.05%)*
	$\rho = 100\%$	493.40 (20.92)	*33.67% (1.43%)*	289.38 (17.38)	*28.86% (1.59%)*	120.19 (11.79)	*27.14% (2.44%)*	23.90 (5.11)	*20.61% (4.12%)*	926.87 (38.40)	*30.61% (1.24%)*
Treatments		***Deceased-Donor Transplants***									
Non-incentivized Treatments		474.77 (20.80)	*32.41% (1.72%)*	380.82 (18.32)	*38.02% (2.29%)*	134.51 (12.06)	*30.43% (3.05%)*	57.23 (7.14)	*49.69% (7.55%)*		
Balanced Incentivized	$\rho = 25\%$	466.92 (21.13)	*31.87% (1.73%)*	384.71 (18.63)	*38.40% (2.33%)*	137.26 (12.19)	*31.05% (3.12%)*	58.44 (7.18)	*50.73% (7.60%)*		
	$\rho = 50\%$	459.08 (21.01)	*31.34% (1.72%)*	388.21 (18.52)	*38.75% (2.32%)*	140.24 (12.24)	*31.72% (3.11%)*	59.80 (7.27)	*51.93% (7.83%)*		
	$\rho = 100\%$	444.56 (21.98)	*30.35% (1.78%)*	395.70 (18.83)	*39.50% (2.38%)*	145.34 (12.26)	*32.87% (3.12%)*	61.73 (7.66)	*53.59% (8.10%)*	1047.33 (28.03)	*34.60% (1.25%)*
Balanced Incentivized also with comp. type X-X pairs	$\rho = 25\%$	467.31 (20.83)	*31.90% (1.71%)*	384.21 (18.51)	*38.35% (2.32%)*	137.35 (12.19)	*31.07% (3.09%)*	58.46 (7.02)	*50.76% (7.58%)*		
	$\rho = 50\%$	458.89 (21.54)	*31.32% (1.73%)*	388.87 (19.07)	*38.82% (2.35%)*	139.91 (12.07)	*31.65% (3.11%)*	59.66 (7.62)	*51.80% (7.98%)*		
	$\rho = 100\%$	444.67 (20.87)	*30.35% (1.68%)*	395.58 (18.31)	*39.49% (2.29%)*	145.46 (12.31)	*32.91% (3.21%)*	61.62 (7.31)	*53.49% (7.81%)*		
Treatments		***Total Transplants***									
Deceased-donor transplantation		474.77 (20.80)	*32.41% (1.72%)*	380.82 (18.32)	*38.02% (2.29%)*	134.51 (12.06)	*30.43% (3.05%)*	57.23 (7.14)	*49.69% (7.55%)*	1047.33 (28.03)	*34.60% (1.25%)*
Living-donor transplantation		748.69 (24.56)	*51.09% (1.90%)*	613.64 (22.63)	*61.24% (2.74%)*	211.32 (15.46)	*47.76% (3.53%)*	80.21 (8.34)	*69.50% (8.39%)*	1653.86 (32.74)	*54.64% (1.49%)*
Regular Exchange		765.47 (24.56)	*52.24% (1.91%)*	659.77 (24.34)	*65.84% (2.86%)*	251.69 (17.03)	*56.88% (3.94%)*	81.13 (8.35)	*70.30% (8.38%)*	1758.06 (35.76)	*58.08% (1.57%)*
Balanced Incentivized	$\rho = 25\%$	807.90 (24.23)	*55.13% (1.98%)*	665.49 (25.29)	*66.41% (2.97%)*	255.24 (17.20)	*57.69% (4.04%)*	82.37 (8.33)	*71.37% (8.37%)*	1811.00 (38.04)	*59.83% (1.64%)*
	$\rho = 50\%$	849.28 (26.15)	*57.96% (2.15%)*	671.15 (25.08)	*66.97% (2.95%)*	258.75 (17.64)	*58.48% (4.09%)*	83.68 (8.48)	*72.52% (8.66%)*	1862.86 (39.48)	*61.54% (1.69%)*
	$\rho = 100\%$	932.47 (27.56)	*63.64% (2.34%)*	683.03 (24.80)	*68.16% (2.94%)*	265.30 (17.34)	*59.96% (4.03%)*	85.61 (8.55)	*74.17% (8.61%)*	1966.41 (42.22)	*64.96% (1.76%)*
Balanced Incentivized also with comp. type X-X pairs	$\rho = 25\%$	812.16 (24.65)	*55.43% (2.01%)*	666.60 (24.62)	*66.52% (2.89%)*	255.66 (17.46)	*57.78% (4.07%)*	82.42 (8.27)	*71.42% (8.42%)*	1816.84 (38.43)	*60.02% (1.66%)*
	$\rho = 50\%$	854.05 (25.78)	*58.28% (2.09%)*	673.92 (25.51)	*67.25% (2.93%)*	258.95 (17.19)	*58.53% (4.03%)*	83.57 (8.83)	*72.41% (8.78%)*	1870.49 (38.50)	*61.79% (1.66%)*
	$\rho = 100\%$	938.07 (26.73)	*64.02% (2.22%)*	684.96 (25.14)	*68.35% (2.90%)*	265.65 (16.90)	*60.05% (4.09%)*	85.52 (8.73)	*74.10% (8.79%)*	1974.20 (42.20)	*65.22% (1.75%)*

Table 10: Simulation results under the de-facto deceased-donor allocation policy for the flow of patients receiving transplant (measured in numbers per year) for different patient blood types. The percentages on the right of each number are the fractions with respect to the new patient flow in the region. Standard errors in 100 simulations are reported in parentheses.

Simulation Results: Average Time to Nonprioritized Deceased-Donor Transplant															
O	**A**	**B**	**AB**	**Overall**	**O**	**A**	**B**	**AB**	**Overall**	**O**	**A**	**B**	**AB**	**Overall**
Deceased-donor transplantation					Balanced inc. $\rho = 25\%$					Balanced inc. also with X-X $\rho = 25\%$				
6.893	5.949	7.235	4.766	6.474	5.491	4.689	5.711	3.799	5.131	5.490	4.686	5.720	3.823	5.132
(0.105)	(0.119)	(0.193)	(0.350)	(0.064)	(0.101)	(0.124)	(0.189)	(0.316)	(0.066)	(0.098)	(0.125)	(0.205)	(0.331)	(0.066)
Living-donor transplantation					Balanced inc. $\rho = 50\%$					Balanced inc. also with X-X $\rho = 50\%$				
6.074	4.928	6.459	3.897	5.585	5.107	4.678	5.667	3.768	4.947	5.111	4.674	5.667	3.758	4.948
(0.103)	(0.126)	(0.196)	(0.288)	(0.068)	(0.103)	(0.128)	(0.175)	(0.292)	(0.065)	(0.098)	(0.125)	(0.189)	(0.306)	(0.066)
Regular exchange					Balanced inc. $\rho = 100\%$					Balanced inc. also with X-X $\rho = 100\%$				
5.932	4.707	5.752	3.837	5.347	4.997	4.699	5.630	3.684	4.895	5.009	4.691	5.620	3.680	4.895
(0.108)	(0.125)	(0.190)	(0.325)	(0.070)	(0.105)	(0.127)	(0.180)	(0.319)	(0.070)	(0.107)	(0.136)	(0.185)	(0.309)	(0.072)

Table 11: Simulation results under the de-facto allocation policy for deceased-donor waiting time for nonprioritized patients from different blood types (measured in years).

Simulation Results: Average Time to Any Type of Transplant															
O	**A**	**B**	**AB**	**Overall**	**O**	**A**	**B**	**AB**	**Overall**	**O**	**A**	**B**	**AB**	**Overall**
Deceased-donor transplantation					Balanced inc. $\rho = 25\%$					Balanced inc. also with X-X $\rho = 25\%$				
6.893	5.949	7.235	4.766	6.474	3.587	2.676	3.265	2.597	3.159	3.548	2.655	3.258	2.599	3.133
(0.105)	(0.119)	(0.193)	(0.350)	(0.064)	(0.107)	(0.111)	(0.157)	(0.261)	(0.066)	(0.101)	(0.106)	(0.161)	(0.274)	(0.065)
Living-donor transplantation					Balanced inc. $\rho = 50\%$					Balanced inc. also with X-X $\rho = 50\%$				
3.862	3.057	4.095	2.743	3.537	3.401	2.626	3.191	2.517	3.051	3.362	2.602	3.175	2.508	3.023
(0.118)	(0.125)	(0.206)	(0.252)	(0.076)	(0.100)	(0.111)	(0.176)	(0.260)	(0.067)	(0.091)	(0.109)	(0.157)	(0.261)	(0.062)
Regular exchange					Balanced inc. $\rho = 100\%$					Balanced inc. also with X-X $\rho = 100\%$				
3.766	2.720	3.331	2.654	3.258	2.942	2.522	3.057	2.386	2.787	2.912	2.504	3.032	2.383	2.762
(0.114)	(0.110)	(0.171)	(0.280)	(0.067)	(0.097)	(0.109)	(0.168)	(0.252)	(0.064)	(0.092)	(0.109)	(0.161)	(0.242)	(0.061)

Table 12: Simulation results under the de-facto deceased-donor allocation policy for average waiting time for any type of transplant for different patient blood types (measured in years).

Appendix D Perfect Matching with (Heterogenous) Tissue-Type Incompatibilities

In this appendix, we study the limit assumptions on the patient types under which different populations of pairs can be matched or patients can be assigned deceased-donor kidneys. The lemmas that we establish below are used in all results regarding steady states of the transplantation policies.

D.1 Matching Deceased-Donor Kidneys

We first consider the case when deceased-donor kidneys are matched with patients. We make the following regularity assumption on the frequency and incompatibility probability of patient types.

Assumption 4 *For every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$, such that for every $k > k_0$ and $l \leq k$ and every permutation σ of patient types,*

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq 1 - \prod_{i=1}^{l} \theta_{\sigma(i),k}.$$

When $\epsilon \to 0$, the regularity assumption can be rewritten as $\sum_{i=l+1}^{k} m_{\sigma(i),k} \geq \prod_{i=1}^{l} \theta_{\sigma(i),k}$. It implies that if you take a set of patients and a set of kidneys with the same measure, then for any

set of patient types the measure of patients with those types is greater than or equal to the measure of the set of kidneys who are tissue-type incompatible with all the other patient types.

Under this assumption, we get the following result.

Lemma 3 *Suppose Assumption 4 holds. Consider a measurable set of patients and deceased-donor kidneys that are blood-type compatible with all the patients such that both sets have the same measure. Suppose these sets are formed randomly using the governing population distributions. Then, as the number of patient types k goes to infinity, almost every patient can be matched with a compatible kidney.*

Proof. Without loss of generality, consider the case when the measures of the two sets are the same and equal to one. Fix a small $\epsilon > 0$. By Assumption 4, there exists k_0 such that, for every $k > k_0$, $l \leq k$, and permutation σ, $\frac{1-\prod_{i=1}^{l} \theta_{\sigma(i),k}}{\sum_{i=1}^{l} m_{\sigma(i),k}} \geq 1 - \epsilon$. Consider any such k.

We use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1 - \epsilon$ measure of the kidneys can be matched with compatible patients. Consider a random measurable subset of patients with measure $1 - \epsilon$. Since the subset is chosen randomly, the compatibility of patients with the kidneys can still be formed randomly using the governing population. We need to show that for any subset of patients, the measure of kidneys that are compatible with at least one patient is weakly greater than the measure of patients. Without loss of generality, instead of considering any set of patients we can consider the set of all patients who have types from any given set. Let the set of patient types be $\{\sigma(1), \ldots, \sigma(l)\}$.

The measure of patients that have a type in this set is $(1 - \epsilon) \sum_{i=1}^{l} m_{\sigma(i),k}$. The measure of kidneys that are incompatible with all such types is $\prod_{i=1}^{l} \theta_{\sigma(i),k}$ because the measure of kidneys is one. Therefore, the measure of kidneys that are compatible with at least one patient in the set is $1 - \prod_{i=1}^{n} \theta_{\sigma(i),k}$. The desired inequality holds by Assumption 4. The claim of the lemma follows by taking the limit as $\epsilon \to 0$ and $k \to \infty$. ■

D.2 Matching Type A-B Pairs with Type B-A Pairs

We next consider the case when we match reciprocal pairs, A-B with B-A. For any such pair, tissue-type compatibility is not known because the pair is blood-type incompatible. Therefore, for any such pair, tissue-type incompatibility is determined randomly as in the overall population.

We make the following assumption on how the market grows, which guarantees that we can match almost every patient in two measurable sets of A-B pairs and B-A pairs that have the same measure.

Assumption 5 *For every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$, $l \leq k$, and every permutation σ of patient types,*

$$(1 - \epsilon) \sum_{i=1}^{l} m_{\sigma(i),k} \leq \sum_{i=1}^{k} m_{\sigma(i),k} [1 - \prod_{j=1}^{l} (1 - (1 - \theta_{\sigma(j),k})(1 - \theta_{\sigma(i),k}))].$$

Consider two measurable sets of A-B and B-A pairs with the same measure. As $\epsilon \to 0$, the assumption guarantees that for any measurable set of reciprocal-type pairs, say B-A, the measure

of this set is smaller than the measure of A-B pairs that are compatible with at least one B-A pair in this set.

Lemma 4 *Suppose Assumption 5 holds. Consider two measurable sets of A-B and B-A pairs that have the same measure. Suppose these sets are formed randomly using the governing population distributions. Then, as the number of patient types k goes to infinity, almost every pair can be matched with a compatible pair.*

Proof. Without loss of generality, consider the case when the measures of the two sets are the same and equal to one. Fix a small $\epsilon > 0$. By Assumption 5, there exists k_0 such that, for every $k > k_0$, $l \leq k$, and permutation σ, $\frac{\sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]}{\sum_{i=1}^{l} m_{\sigma(i),k}} \geq 1-\epsilon$. Consider any such k.

Like before, we use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1-\epsilon$ measure of the B-A pairs can be matched with compatible A-B pairs. Consider a random measurable subset of B-A pairs with measure $1-\epsilon$. Since the subset is chosen randomly, the compatibility of donors with patients can still be formed randomly using the governing population. We need to show that for any subset of B-A pairs, the measure of A-B pairs who are compatible with at least one B-A pair in the chosen set is weakly greater than the measure of the chosen set of B-A pairs. Without loss of generality, instead of considering any set of B-A pairs, we can consider the set of all B-A pairs with patients that have types from any given set. Let the set of patient types be $\{\sigma(1), \ldots, \sigma(l)\}$.

The measure of B-A pairs with patients that have a type in this set is $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}$. The measure of A-B pairs with patient type $\sigma(i)$ who are incompatible with all such pairs is $m_{\sigma(i),k}\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))$. Therefore, the measure of A-B pairs with patient type $\sigma(i)$ who are compatible with at least one B-A pair from the chosen set is $m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. Hence, the measure of A-B pairs that are compatible with at least one B-A pair in the chosen set is $\sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. This sum is greater than the measure of chosen B-A pairs, $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}$, by Assumption 5.

Therefore, $1-\epsilon$ measure of B-A pairs can be matched with compatible A-B pairs. The lemma follows by taking $\epsilon \to 0$ and $k \to \infty$. ■

D.3 Matching Overdemanded-Type Pairs Except A-B Pairs with Underdemanded-Type Pairs Except B-A Pairs

We next consider the case when we match overdemanded-type pairs except A-B pairs with underdemanded-type pairs except B-A pairs. In the rest of this subsection, when we mention overdemanded-type pairs we exclude A-B pairs, and similarly, when we mention underdemanded-type pairs we exclude B-A pairs.

We make the following assumption on the frequency and incompatibility probability of patient types.

Assumption 6 *For every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$, $l \leq k$, $0 \leq \rho \leq 1$, and every permutation σ of patient types,*

$$(1-\epsilon)\sum_{i=1}^{l}\frac{m_{\sigma(i),k}(\theta_{\sigma(i),k}+\rho(1-\theta_{\sigma(i),k}))}{M}\leq\sum_{i=1}^{k}m_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))],$$

where $M=\sum_{i=1}^{k}m_{\sigma(i),k}(\theta_{\sigma(i),k}+\rho(1-\theta_{\sigma(i),k}))$.

For overdemanded-type pairs, only tissue-type incompatible ones participate in the regular exchange. However, in the incentivized exchange, compatible pairs also participate. As a result, a fraction of the overdemanded pairs are compatible, while the rest are incompatible. Here, ρ is the participation rate of compatible pairs. The assumption guarantees that, for any set of overdemanded-type pairs, the set of underdemanded pairs that are compatible with at least one pair in the set has a greater measure as $\epsilon\to 0$.

Lemma 5 *Suppose Assumption 6 holds. Consider two measurable sets of overdemanded X-Y pairs and underdemanded Y-X pairs with the same measure. Suppose that a fraction of overdemanded X-Y pairs are known to be tissue-type incompatible and the rest are known to be tissue-type compatible, but otherwise these sets are formed randomly using the governing population distributions. Then, as the number of patient types k goes to infinity, almost every pair can be matched with a compatible pair.*

Proof. Without loss of generality, consider the case when the measures of the two sets are the same and equal to one. Then, for underdemanded Y-X pairs, $m_{i,k}$ measure of the patients have type i for every i. For overdemanded X-Y pairs, some are known to be tissue-type compatible while others are tissue-type incompatible. The measure of compatible pairs is proportional to $\rho m_{i,k}(1-\theta_{i,k})$ and the measure of incompatible pairs is proportional to $m_{i,k}\theta_{i,k}$. Therefore, the measure of overdemanded X-Y pairs with patient type i is $\frac{m_{\sigma(i),k}(\theta_{\sigma(i),k}+\rho(1-\theta_{\sigma(i),k}))}{M}$ where $M=\sum_{i=1}^{k}m_{\sigma(i),k}(\theta_{\sigma(i),k}+\rho(1-\theta_{\sigma(i),k}))$.

Fix a small $\epsilon>0$. Consider any k that satisfies Assumption 6. Like before, we use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1-\epsilon$ measure of the overdemanded X-Y pairs can be matched with compatible underdemanded Y-X pairs. Consider a random measurable subset of overdemanded X-Y pairs with measure $1-\epsilon$. Since the subset is chosen randomly, the compatibility of pairs can still be formed randomly using the governing population. We need to show that, for any subset of overdemanded X-Y pairs, the measure of underdemanded Y-X pairs who are compatible with at least one overdemanded X-Y pair is weakly greater than the measure of overdemanded X-Y pairs. In this calculation, we use a lower bound for the measure of such underdemanded Y-X pairs by assuming that if their patient has type i, then they are incompatible with overdemanded X-Y pairs with patient type i. Without loss of generality, instead of considering any set of overdemanded X-Y pairs, we can consider the set of all overdemanded X-Y pairs with patients who have tissue types from any given set. Let the set of patient types be $\{\sigma(1),\ldots,\sigma(l)\}$.

The measure of overdemanded X-Y pairs with patients who have types in the set is $(1-\epsilon)\sum_{i=1}^{l}\frac{m_{\sigma(i),k}(\theta_{\sigma(i),k}+\rho(1-\theta_{\sigma(i),k}))}{M}$. The measure of underdemanded Y-X pairs with patient type $\sigma(i)$ for $i\leq l$ who are incompatible with all such pairs is $m_{\sigma(i),k}\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))$. Note that we are assuming that these pairs are incompatible with overdemanded X-Y pairs with patient

of type $\sigma(i)$. On the other hand, if $i > l$, then the measure of underdemanded Y-X pairs with patient type $\sigma(i)$ who are incompatible with all such pairs is $m_{\sigma(i),k} \prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))$. Hence, the measure of underdemanded Y-X pairs that are compatible with at least one overdemanded X-Y pair in the chosen set is at least $\sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. This sum is greater than the measure of chosen overdemanded X-Y pairs Assumption 6.

The proof that $1-\epsilon$ measure of overdemanded X-Y pairs can be matched follows. The lemma follows by taking $k \to \infty$ and $\epsilon \to 0$. ■

D.4 Matching Self-Demanded-Type Pairs

In this section, we consider the case when we match self-demanded type pairs. Fix any self-demanded-type pair X-X for some blood type X. Any such pair in the exchange pool is tissue-type incompatible. We match these pairs with each other. Therefore, in contrast with the previous sections, this is a one-sided matching problem.

We make the following assumption to show that almost every pair can be matched in the limit.

Assumption 7 *For every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$, $l \leq k$, and every permutation σ of patient types,*

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}\theta_{\sigma(i),k} \leq \sum_{i=1}^{k} m_{\sigma(i),k}\theta_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))].$$

Our next result shows that under this assumption almost all self-demanded pairs can be matched.

Lemma 6 *Suppose Assumption 7 holds. Consider a set of self-demanded-type pairs X-X that are tissue-type incompatible. Assume that this set is formed randomly using the governing population distributions. Then, as the number of patient types k goes to infinity, almost every pair can be matched with a compatible pair.*

Proof. Since the pairs are tissue-type incompatible, but otherwise formed randomly using the governing population distributions, for each patient type i, the measure of pairs with patient type i is proportional to $m_i\theta_i$.

Fix a small $\epsilon > 0$. Consider any k that satisfies Assumption 7.

We use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1-\epsilon$ fraction of the self-demanded X-X pairs can be matched with compatible self-demanded X-X pairs. To show this, we first construct a two-sided matching problem with these pairs. For any patient type i, we split the set of pairs with patient type i into two sets with equal measure. These sets are then added to different sides of the market. As a result, we get a two-sided matching problem where each side has X-X pairs where those with patient type i have a measure proportional to $m_i\theta_i$. For ease of exposition, suppose that the measure is exactly $m_i\theta_i$.

Consider one side of the market. To apply Gale's Supply-Demand Theorem, take a random measurable subset of pairs on this side of the market that has measure $1-\epsilon$ fraction of all pairs on this side. Since the subset is chosen randomly, the compatibility of patients can still be formed randomly using the governing population. We need to show that for any subset of pairs, the measure

of pairs on the other side of the market that are compatible with at least one pair in the set is weakly greater than the measure of chosen pairs. Without loss of generality, instead of considering any set of patient types, we can consider the set of all patients that have types from any given set. Let this set be $\{\sigma(1), \ldots, \sigma(l)\}$.

The measure of the set of pairs that have patient types from this set is $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}\theta_{\sigma(i),k}$. The measure of pairs that have patient type $\sigma(i)$ on the other side that are incompatible with all such types is $m_{\sigma(i),k}\theta_{\sigma(i),k}\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))$. The measure of pairs that have patient type $\sigma(i)$ on the other side that are compatible with at least one type in the set is $m_{\sigma(i),k}\theta_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. Therefore, the measure of pairs on the other side that are compatible with at least one pair in the chosen set is $\sum_{i=1}^{k} m_{\sigma(i),k}\theta_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. This sum is greater than the measure of pairs that are chosen, which is $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}\theta_{\sigma(i),k}$ by Assumption 7.

Therefore, $1-\epsilon$ fraction of pairs on both sides of the market can be matched. As we take $\epsilon \to 0$ and $k \to \infty$, we establish the desired result that almost every pair is matched with a compatible pair. ■

D.5 Sufficient Limit Conditions

In the next lemma, we provide sufficient conditions under which all of the limit assumptions hold.

Lemma 7 *Suppose that $\theta_{i,k} = \theta < 1$ and $m_{i,k} \to 0$ for every $i \leq k$ as $k \to \infty$. Then Assumptions 4, 5, 6, and 7 hold.*

Proof. When $\theta_{i,k} = \theta$ for every $i \leq k$, Assumption 4 reduces to

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq 1-\theta^l$$

under the same conditions as stated therein. Likewise, Assumption 5 reduces to

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq 1-(1-(1-\theta)^2)^l,$$

and Assumptions 6 and 7 reduce to

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq \sum_{i=1}^{l} m_{\sigma(i),k}[1-(1-(1-\theta)^2)^{l-1}] + \sum_{i=l+1}^{k} m_{\sigma(i),k}[1-(1-(1-\theta)^2)^l].$$

If we show that $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq \sum_{i=1}^{l} m_{\sigma(i),k}[1-\beta^{l-1}] + \sum_{i=l+1}^{k} m_{\sigma(i),k}[1-\beta^l]$ for every $\beta < 1$ under the conditions stated in these assumptions, then we will be done. This inequality can be rewritten as

$$(\beta^{l-1}-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq \sum_{i=l+1}^{k} m_{\sigma(i),k}[1-\beta^l]. \tag{23}$$

For a fixed ϵ such that $1 > \epsilon > 0$, there exists a natural number n such that $\beta^{n-1} \geq \epsilon > \beta^n$. Then Inequality 23 holds for $l > n$ for every k because the left side of the inequality is negative

whereas the right side is positive. Furthermore, as $k \to \infty$ Inequality 23 holds also for every $l \leq n$ because $m_{i,k} \to 0$ for every i and n is a fixed natural number which does not depend on k. In this case, the left side converges to zero and the right side is always positive. ■