A Lot of Ambiguity*

Zvi Safra† Uzi Segal‡

March 31, 2020

Abstract

We consider a risk averse decision maker who dislikes ambiguity as in the Ellsberg urns. We analyze attitudes to ambiguity when the decision maker is exposed to unrelated sequences of ambiguous situations. We discuss the Choquet expected utility, the smooth, and the maxmin models. Our main results offer conditions under which ambiguity aversion disappears and conditions under which it does not.

Keywords: Ellsberg urns, repeated ambiguity, repeated risk, Choquet expected utility, maxmin, the smooth model

1 Introduction

A patient suffers from a certain disease. The doctor offers two possible treatments. A standard, well investigated treatment Y, which with probability p leads to a good outcome and with probability $1-p$ leads to a less favorable outcome, which is still better than no treatment.[1] Alternatively, she offers him a new treatment L with somewhat ambiguous probabilities of success. It is however known that whatever the outcome, it improves over that of the standard treatment. Moreover, although the probabilities are not known for sure, they are believed to be somewhere around $p : 1-p$. Let X denote the

*Acknowledgments to be added.

†Warwick Business School (zvi.safra@wbs.ac.uk) and Emeritus professor, Tel Aviv University.

‡Department of Economics, Boston College (segalu@bc.edu).

[1] For simplicity, suppose that all outcomes are monetary payoffs, for example, the disease only effects people's ability to work.

probabilistic lottery yielding the outcomes of L with the believed probabilities and assume that the expected value of X is zero. Note that X dominates Y by first-order stochastic dominance. Both treatments are preferred to no treatment and the question is which of the two to choose. The patient is ambiguity averse, and as the improvement in the outcomes of the new treatment is not much, he prefers the old treatment with the known probability of success. In other words, $Y \succ L$.

The doctor does not have any information she did not share with the patient. Moreover, although she knows that she will see many patients like him, she believes that she won't gain any information about the probability of success of the new treatment, as this probability depends entirely on unobservable characteristics of the patients. Her preferences over risk and uncertain prospects are the same as the patient's (alternatively, she adopts the patient's preferences). Does it follow that she too will prefer the standard treatment to the new one?

Although they have exactly the same information and preferences, there is one dimension in which the patient and the doctor are different, and this is the number of cases they face. The patient sees only one case, his. Ambiguity aversion can be explained as fear of the unknown. Many people believe that they are unlucky and therefore, if they choose the ambiguous prospect, they'll find out that the winning probabilities took a bad turn and are on the lower side of their expectations. But such people do not necessarily believe that they are always unlucky. Thus the doctor is ambiguity averse, but as she is facing many similar cases, her aversion to each case may diminish. Furthermore, this may lead her to prefer the new treatment over the standard one.

In this paper we formalize this discussion. Suppose that the doctor has to make a decision for $n > 1$ (identical) people. Denote n repetitions of the standard treatment by Y^n and of the new treatment by L^n. Both are better than no treatment. We show that under some conditions, and for sufficiently large n, $L^n \succ Y^n$. That is, n repetitions of the new treatment are, eventually, preferred over n repetitions of the standard treatment (Theorem 1).

Next consider an alternative scenario involving, again, a patient and a doctor. This time, avoiding treatment does not lead to a bad outcome but may be costly, and only the ambiguous treatment L is available. Suppose that the probabilistic lottery X, yielding the outcomes of L with the believed probabilities $p : 1 - p$, has a positive expected value and that X and all its repetitions X^n are preferred to no treatment, no matter how small is its cost, while no treatment is preferred to L. Can it be the case that eventually L^n becomes desirable? We show that this is indeed the case. Under some conditions, n repetitions of the ambiguous treatment are eventually preferred to no treatment (Theorem 2).

Should society encourage, maybe even enforce, the use of the ambiguous treatment? Patients may be willing to pay the extra price for the unambiguous treatment if it exists, or to bear the cost of no treatment if an alternative treatment does not exist. But if society adopts the point of view of social planers and care takers (even if they do not have any better information), then it may opt out for the ambiguous treatment. Providing general answers to such questions is beyond the scope of the current paper but our aim is to show that, at least under some conditions, such questions are not meaningless.

Theorems 1 and 2 of section 3 analyze Choquet expected utility preferences (Schmeidler [27]). Under some conditions, similar results hold in the smooth recursive utility model (Klibanoff, Marinacci, and Mukerji [15]), but under some other conditions they do not hold (Theorem 3 in Section 4). On the other hand, in the maxmin expected utility model (Gilboa and Schmeidler [13]) similar results hold only under some extreme conditions (Theorem 4 in Section 5). We discuss some further issues and the literature in section 6. All claims are proved in the appendixes.

2 Setup

One ball is picked at random out of an urn containing Γ balls of γ colors. Let s_i be the state of nature "color i is picked." Denote $S = \{s_1, \ldots, s_\gamma\}$, and define $\Sigma = 2^S$. The number of balls of some colors may be known to be Γ/γ, making the corresponding states of nature probabilistic with probability $\frac{1}{\gamma}$. This ratio also serves as an anchor for non probabilistic states and events. For example, in the 3-color Ellsberg [4] urn which contains 90 balls, of which 30 are red and each of the other 60 is either black or yellow, the anchoring probabilities are $\frac{1}{3}$ for each of the three colors and $\frac{2}{3}$ for each of complementing events.[2] For more on the anchoring probabilities, see Fox and Tversky [11],

[2]More complicated urns are also possible, for example, an urn containing 100 balls. Twenty of which are yellow, and each of the others is either red or green. The anchoring probabilities for (Y,R,G) are $(\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$, but this situation can easily be described as an urn containing balls of five colors.

Nau [21], Chew and Sagi [3], and Ergin and Gul [7]. For $E = \{s_{i_1}, \ldots, s_{i_\ell}\} \in \Sigma$, let $P(E) = \frac{\ell}{\gamma}$.

Assume now the existence of a sequence of such urns. Let $S_i = S$ be the set of states in urn i with the corresponding algebra $\Sigma_i = \Sigma$. The information regarding each of these urns is the same. Moreover, the outcome, or even the mere existence of any urn doesn't change the decision maker's information regarding any other urn. Let $\mathcal{S}^n = S_1 \times \ldots \times S_n$ and $\Omega^n = 2^{\mathcal{S}^n}$ (note that $\Omega^1 = \Sigma$). For $E \in \Omega^n$, define $P^n(E)$ to be the number of sequences in E divided by γ^n.

Consider a non-degenerate act $L = (x_1, E_1; \ldots; x_m, E_m)$ where $x_1, \ldots, x_m \in \Re$, $x_1 < \ldots < x_m$, and $E_1, \ldots, E_m$ is a partition of Σ. The outcomes $x_1, \ldots, x_m$ denote departures from the current wealth level, which is assumed throughout to be fixed. Define the anchor lottery $X = (x_1, p_1; \ldots;\, x_m, p_m)$ where $p_i = P^1(E_i) := P(E_i)$ is the anchor probability of E_i and denote its expected value by $\mathrm{E}(X)$. The act L^n is the sequence of act L played once on each of the n urns. We assume that the decision maker is interested in the total sum of outcomes he wins but not in the order or the composition of colors leading to these wins and will therefore view L^n as $(x_1^n, E_1^n; \ldots; x_{k_n}^n, E_{k_n}^n)$, where $x_1^n = nx_1 < \ldots < x_{k_n}^n = nx_m$ and E_i^n is the collection of sequences of events from $\Sigma_1, \ldots, \Sigma_n$ such that the sum of their corresponding outcomes is x_i^n. The lottery $X^n = (x_1^n, p_1^n; \ldots; x_{k_n}^n, p_{k_n}^n)$ is a sequence of n independent lotteries of type X where p_i^n is the anchor probability $P^n(E_i^n)$. The lottery X^n serves as a natural anchor for L^n.

Consider a decision maker with preferences $\succeq^n$ over $\mathcal{L}^n$, the space of all real acts over Ω^n. We assume that the decision maker evaluates lotteries with known probabilities using expected utility theory with the twice differentiable vNM function u. We assume that the decision maker is risk averse (hence his vNM utility u is concave) and ambiguity averse in the sense that he prefers playing X^n to playing L^n. Finally, we assume throughout that $\lim\limits_{x \to -\infty} \frac{u''(x)}{u'(x)}$ exists, but not necessarily that it is finite.

3 Choquet Expected Utility

In this section we consider preferences over ambiguous prospects that can be represented by the Choquet expected utility (CEU) model (Schmeidler [27]). According to this theory, there are capacities $\nu^n : \Omega^n \to [0, 1]$ such that

$\nu^n(\varnothing) = 0$, $\nu^n(\mathcal{S}^n) = 1$, and the value of L^n, $\text{CEU}^n(L^n)$, is

$$u(x^n_{k_n})\nu^n(E^n_{k_n}) + \sum_{i=1}^{k_n-1} u(x^n_i)\left[\nu^n\left(\bigcup_{j=i}^{k_n} E^n_j\right) - \nu^n\left(\bigcup_{j=i+1}^{k_n} E^n_j\right)\right] \tag{1}$$

To ensure ambiguity aversion we assume that $\nu^n(E) \leqslant P^n(E)$ for all $E \in \Omega^n$, which is equivalent to $P^n \in \text{Core}(\nu^n)$.[3]

Ambiguity aversion permits the union of two ambiguous events to be non-ambiguous. For example, in the 3-color Ellsberg urn, the union of the two ambiguous colors leads to an event with probability $\frac{2}{3}$. The contribution of an event to the value of a lottery can therefore be larger than its anchor probability. If there is only a finite number of events, then there is of course an upper bound to the ratio between the contribution of the capacities generated by all events and their anchor probabilities. Our main requirement is that the following boundedness condition holds uniformly for all n, that is, that the potential over-estimation of the contribution of all events will not go to infinity. Formally:

Boundedness There is K such that for all n and for all disjoint events $E, E' \in \Omega^n$, $\nu^n(E \cup E') - \nu^n(E) \leqslant KP^n(E')$.

This condition is satisfied in a trivial way if the capacity is a probability measure. The following is an example of non-probabilistic capacities that satisfy boundedness.

Example 1 Assume urns with 100 balls each of two colors, G and R. When there are n urns, there are 2^n possible outcomes of the samples (that is, $\{G, R\}^n$), with typical elements $t = (t_1, \ldots, t_n)$, where for all i, $t_i \in \{G, R\}$. Recall that the anchor probability P^n of each event E is $|E|/2^n$. Define capacities ν^n by

$$\nu^n(E) = \begin{cases} 0 & P^n(E) \leqslant \frac{1}{2} \\ \\ 2P^n(E) - 1 & \text{otherwise} \end{cases}$$

By definition, $\nu^n(E \cup E') - \nu^n(E) \leqslant \frac{|E'|}{2^{n-1}} = 2P^n(E')$, hence these capacities are bounded with $K = 2$.

[3]The core of a capacity ν is the set of all probability distributions q such that for all E, $q(E) \geqslant \nu(E)$.

Note that this is a product capacity. For all $E = E^1 \times \ldots \times E^n$, $\nu^n(E) = \prod_{i=1}^n \nu^1(E^i) = 0$, unless for all i, $E^i = \{G, R\}$, in which case $\nu^n(E) = \prod_{i=1}^n \nu^1(E^i) = 1$. ■

Following the discussion in the introduction, consider a given ambiguous act L with the anchor lottery X. Suppose that the expected value of X is zero and let X dominate a lottery Y by first order stochastic dominance (FOSD). Theorem 1 shows that as $n \to \infty$, the decision maker will prefer playing L for n times (that is, L^n) rather than playing Y for n times.

Theorem 1 Suppose that the CEU preferences satisfy ambiguity aversion, risk aversion, and boundedness. Let L be an ambiguous act with an anchor lottery X such that $\mathrm{E}(X) = 0$. Then for every Y dominated by X by strict FOSD there exists n^* such that for all $n \geqslant n^*$, $L^n \succ Y^n$.

Remark: The proof of Theorem 1 covers also the case $\mathrm{E}(X) < 0$, except for the case where $\lim\limits_{x \to -\infty} u'(x) = \infty$ but $\lim\limits_{x \to -\infty} \frac{u''(x)}{u'(x)} = 0$.

Consider now a different case, where $\mathrm{E}(X) > 0$. This of course doesn't mean that the decision maker accepts X, or even that if he accepts it once he would accept it n times. And it may certainly happen that he will accept X, but will decline the corresponding ambiguous act L. For example, the decision maker may accept the lottery $(-100, \frac{1}{2}; 110, \frac{1}{2})$, yet decline the act where in the two-color Ellsberg urn he wins 110 if he correctly guesses the color of the drawn ball, but loses 100 if he does not. By continuity, there are lotteries Y dominated by 0 which are preferred to L. Nevertheless, if for a sufficiently large n, $X^n \succeq 0$, then for any lottery Y dominated by 0, the decision maker prefers L^n to Y^n for a sufficiently large n .

Theorem 2 Suppose that the CEU preferences satisfy ambiguity aversion, risk aversion, and boundedness. Let L be an ambiguous act with an anchor lottery X such that $\mathrm{E}(X) > 0$. If there exists n_0 such that for all $n \geqslant n_0$, $X^n \succeq 0$, then for every Y dominated by 0 by strict FOSD, there exists n^* such that for all $n \geqslant n^*$, $L^n \succ Y^n$.

Stricter results can be obtained with further restrictions on the utility function u and on the lottery X. Assume first that u is bounded from above, which is used to avoid phenomena in the spirit of the St. Petersburg paradox.

Proposition 1 shows that under these conditions, from a certain point on the ambiguous acts L^n become strictly desirable.[4]

Proposition 1 Suppose that the CEU preferences satisfy ambiguity aversion, risk aversion, and boundedness and suppose that u is bounded from above. Let L be an ambiguous act with an anchor lottery X such that $\mathrm{E}(X) > 0$. If there exists $\varepsilon > 0$ and n_0 such that for all $n \geqslant n_0$, $X^n \succeq n\varepsilon$, then there exists n^* such that for all $n \geqslant n^*$, $L^n \succ 0$.

Assuming exponential or linear u (thus representing constant absolute risk aversion), the next proposition strengthens Theorems 1 and 2 to general acts L, regardless of the expectation of the anchor lottery X.

Proposition 2 Suppose that the CEU preferences satisfy ambiguity aversion, constant absolute risk aversion, and boundedness. Then for every $Y \prec X$ there exists n^* such that for all $n \geqslant n^*$, $L^n \succ Y^n$.

How restrictive is the boundedness assumption? That is, does boundedness imply that ν^n converge to a capacity ν with a degenerate core, which is equal to the anchor probability measure? If this is the case, then the boundedness assumption makes the analysis trivial, because the limit of the capacities ν^n is just the anchor probability vector and CEU becomes EU. We show however that this is not the case. There are bounded capacities for which the cores do not converge to a singleton.

Example 1 (contd.) For $s \in \mathcal{S}^n$, define

$$\tilde{P}^n(s) = \begin{cases} 0 & |\{i : s_i = G\}| < \frac{n}{2} \\ 0 & |\{i : s_i = G\}| = \frac{n}{2} \text{ and } s_1 = G \\ \frac{1}{2^{n-1}} & \text{otherwise} \end{cases}$$

For each $E \in \Omega^n$, define $\tilde{P}^n(E) = \sum_{s \in E} \tilde{P}^n(s)$. For every E,

$$\tilde{P}^n(E) \geqslant 2 \max\left\{\frac{|E|}{2^n} - \frac{1}{2}, 0\right\} = \nu^n(E)$$

[4] A sufficient condition for boundedness from above is that the Arrow-Pratt measure of absolute risk aversion is bounded away from 0. That is, that there exists $\delta > 0$ such that for all z, $r_u(z) = -u''(z)/u'(z) > \delta$. To see it, let $v(z) = -e^{-\delta z}$. Then $r_u(z) > r_v(z)$ and, by Pratt [23], there exists a concave h such that $u = h \circ v$. The boundedness of u follows from that of v.

Hence $\tilde{P}^n$ is in the core of ν^n and clearly $\tilde{P}^n$ and P^n do not converge to the same limit. ■

Our results do not always hold without the boundedness assumption. See example 2 in the appendix. The boundedness of u from above is required for Proposition 1. See example 3 in the appendix.

4 The Smooth Model

Klibanoff, Marinacci, and Mukerji [16] suggested the following smooth case of the recursive model [28]. According to their model, the decision maker has a set of possible probability distributions, and he attaches a probability to each of them. He computes the certainty equivalent of the uncertain act using expected utility with the vNM function u for each of the possible distributions, and then evaluates the lottery over these values using the vNM function ϕ.[5] Ambiguity aversion in this model is reflected by ϕ being more concave than u. Ambiguity neutrality is obtained when ϕ and u are the same.

Formally, let $L = (x_1, E_1; \ldots; x_m, E_m)$ be an ambiguous act. The decision maker believes that with probability μ^i, $i = 1, \ldots, \ell$, the probability distribution of L is given by $p^i = (p^i_1, \ldots, p^i_m)$. Denote $X_{p^i} = (x_1, p^i_1; \ldots; x_m, p^i_m)$ and let $p = \sum_{i=1}^{\ell} \mu^i p^i$. Hence, $X = (x_1, p_1; \ldots; x_m, p_m) = \sum_{i=1}^{\ell} \mu^i X_{p^i}$ is the anchor lottery of L. The value of L under the smooth model is given by[6]

$$\mathrm{SM}^{\phi u}(L) = \sum_{i=1}^{\ell} \mu^i \cdot \phi \circ u^{-1} \left(\mathrm{EU}^u(X_{p^i})\right)$$

When there is no ambiguity (that is, the decision maker believes that with probability 1 the probability distribution associated with L is p), then the value of L is $\phi \circ u^{-1}(\mathrm{EU}^u(X))$ which represents the same order as EU with the vNM utility u. Note that $\mathrm{EU}^u(X)$ is the value attached to L by an ambiguity neutral decision maker for whom $\phi = u$. To see why, observe that

$$\mathrm{SM}^{uu}(L) = \sum_{i=1}^{\ell} \mu^i \cdot \mathrm{EU}^u(X_{p^i}) = \mathrm{EU}^u \left(\textstyle\sum_{i=1}^{\ell} \mu^i X_{p^i}\right) = \mathrm{EU}^u(X)$$

[5] The original paper [16] denoted this function v.

[6] Since this model is using two different vNM functions, we add a superscript index (u or ϕ) to indicate the utility function used in the EU operator.

As before, let X^n and L^n be n-repetitions of X and L. The value of X^n is $\mathrm{EU}^u(X^n)$. Consider L^n. A typical sequence in L^n is a list of n lotteries, each taken from the set $\{X_{p^1}, \ldots, X_{p^\ell}\}$, where X_{p^i} appears j_i times, $i = 1, \ldots, \ell$, and $\sum_i j_i = n$. The probability of such a sequence is the product of the corresponding μ^i probabilities, that is, $\prod_i (\mu^i)^{j_i}$. There are $(\ell)^n$ (ℓ to the power of n) such possible sequences, denote them $\{Y_j^n\}_{j=1}^{(\ell)^n}$ and denote their corresponding probabilities μ_j^n. We thus obtain that

$$\mathrm{SM}^{\phi u}(L^n) \quad = \quad \sum_{j=1}^{(\ell)^n} \mu_j^n \cdot \phi \circ u^{-1}(\mathrm{EU}^u(Y_j^n)) \tag{2}$$

The next theorem shows that the results of Theorem 1 hold if the absolute measures or risk aversion of u and ϕ converge to the same limit as $x \to -\infty$. Observe that although $\frac{\phi''(x)}{\phi'(x)} \equiv \frac{u''(x)}{u'(x)}$ implies that ϕ is an affine transformation of u, the restriction $\lim\limits_{x\to-\infty} \frac{\phi''(x)}{\phi'(x)} = \lim\limits_{x\to-\infty} \frac{u''(x)}{u'(x)}$ does not imply that in the limit ϕ is an affine transformation of u. For example, let $u(x) = x$ and $\phi(x) = x^3$ for $x < -1$.

Theorem 3 Suppose that the SM preferences satisfy ambiguity and risk aversion. Let L be an ambiguous act with an anchor lottery X such that $\mathrm{E}(X) = 0$. If $\lim\limits_{x\to-\infty} \frac{\phi''(x)}{\phi'(x)} = \lim\limits_{x\to-\infty} \frac{u''(x)}{u'(x)}$, then for every Y dominated by X by strict FOSD there exists n^* such that for all $n \geqslant n^*$, $L^n \succ Y^n$.

Proposition 1 analyzed conditions under which, within the CEU model, the acts L^n become strictly desirable. The next proposition offers conditions for a similar result under the SM model. For this, we restrict attention to the case where u represents constant absolute risk aversion. Observe that by risk aversion, $X \succ 0$ implies that $\mathrm{E}(X) > 0$.

Proposition 3 Suppose that the SM preferences satisfy ambiguity aversion and constant absolute risk aversion. If $\lim\limits_{x\to-\infty} \frac{\phi''(x)}{\phi'(x)} = \lim\limits_{x\to-\infty} \frac{u''(x)}{u'(x)}$, then for every ambiguous act L with an anchor lottery $X \succ 0$ there exists n^* such that for all $n > n^*$, $L^n \succ 0$.

Theorem 3 and Proposition 3 assume that $\lim\limits_{x\to-\infty} -\frac{\phi''(x)}{\phi'(x)} = \lim\limits_{x\to-\infty} -\frac{u''(x)}{u'(x)}$. The next proposition shows the necessity of this condition. We say that the risk aversion of utility function u is bounded from above [from below] by ζ if for all x, $-u''(x)/u'(x)$ is less than [more than] ζ. The next result shows that if the degree of risk aversion of ϕ is bounded from below by $t > 0$, then for u with degree of risk aversion that is bounded from above by a sufficiently small s^*, if Y is sufficiently close to X then $Y^n \succ L^n$, even if $Y \prec X$.

Proposition 4 Let the SM preferences satisfy ambiguity and risk aversion such that the risk aversion of ϕ is bounded from below by $t > 0$. For every ambiguous act L with anchor lottery X such that $\mathrm{E}(X) = 0$ there is $s^* > 0$ and a neighborhood $\mathcal{N}$ of X such that if the risk aversion of u is bounded from above by s^*, then for every $Y \in \mathcal{N}$ there is n^* such that for all $n > n^*$, $Y^n \succ L^n$.

The next proposition shows that if u represents constant absolute risk aversion and ϕ represents a higher degree or risk aversion, then for each ambiguous lottery L, regardless of the expected value of its probabilistic anchor X, for Y which is sufficiently close to X, $Y^n \succ L^n$, even if $Y \prec X$.

Proposition 5 Let the SM preferences satisfy ambiguity aversion and constant absolute risk aversion with parameter s. If ϕ is bounded from below by $t > s$, then for every ambiguous act L with anchor lottery X there is a neighborhood $\mathcal{N}$ of X such that for every $Y \in \mathcal{N}$ there is n^* such that for all $n > n^*$, $Y^n \succ L^n$.

5 Maxmin Expected Utility

Gilboa and Schmeidler [13] suggested the following maxmin expected utility (MEU) theory. Under ambiguity, the decision maker behaves as if he has a (convex) set of possible probability distributions as well as a utility function u. For each act he computes the expected utility of u with respect to the different possible probability distributions, and evaluates the act as the minimum of these values.

As in the previous chapters, let $L = (x_1, E_1; \ldots; x_m, E_m)$ be an ambiguous act and denote the set of possible probability distributions by Q, with typical elements of the form $q = (q_1, \ldots, q_m)$. As before, $X = (x_1, p_1; \ldots; x_m, p_m)$ is the anchor lottery associated with L. Denoting $X_q = (x_1, q_1; \ldots; x_m, q_m)$, the value of L under the maxmin model is given by

$$\mathrm{MEU}(L) = \min_{q \in Q} \mathrm{EU}\left(X_q\right)$$

An event E is ambiguous if the decision maker may treat it differently from its anchor probability. This means that if the decision maker is ambiguity averse, then the anchor probability $P^1(E)$ is not the minimal possible value of the range of the possible probabilities of E. To see why, note that if L is not a probabilistic act, then there must be at least two ambiguous events in its support. Therefore, there is a lottery $X_{\hat{q}}$ that is dominated by X by FOSD. By definition, $\mathrm{MEU}(L) \leqslant \mathrm{EU}(X_{\hat{q}}) < \mathrm{EU}(X)$.

Consider $L^n = \left(x_1^n, E_1^n; \ldots ; x_{k_n}, E_{k_n}^n\right)$ and the corresponding anchor lottery $X^n = \left(x_1^n, p_1^n; \ldots ; x_{k_n}^n, p_{k_n}^n\right)$ where $p_j^n = P^n(E_j^n)$. As is standard in the literature, we assume that the set of possible priors for L^n is $Q^n = Q \times \ldots \times Q$ and define

$$\mathrm{MEU}(L^n) = \min_{q^n \in Q^n} \mathrm{EU}(X_{q^n}^n)$$

As the lottery $(X_{\hat{q}})^n$ is possible under this set of priors, it follows that the priors in Q^n that minimize the value of L^n must yield a value that cannot exceed the value of $(X_{\hat{q}})^n$.

Suppose that the decision maker is extremely risk averse, in which case his evaluation of a lottery will be close to his evaluation of its worst outcome. Since L^n cannot be inferior to its worst outcome nx_1, it follows that such a decision maker will be almost indifferent between X^n and L^n. Let $Y = X - \varepsilon$ for some $\varepsilon > 0$. Since the worst outcome of Y^n is $n(x_1 - \varepsilon)$, an extremely risk averse person will eventually prefer L^n to Y^n. Theorem 4 formalizes this argument, and shows that this is the only case in which the repeated ambiguous act L^n becomes superior to every such Y. Otherwise, the extreme level of ambiguity aversion generated by the maxmin model will keep L^n less desirable than Y^n for a sufficiently small ε.

Theorem 4 Let the MEU preferences satisfy ambiguity and risk aversion and let L be an ambiguous act with an anchor lottery X such that $\mathrm{E}(X) = 0$.

1. If $\lim\limits_{x \to -\infty} \frac{u''(x)}{u'(x)} = \infty$, then for every $Y = X - \varepsilon$, $\varepsilon > 0$, there exists n^* such that for all $n \geqslant n^*$, $L^n \succ Y^n$.
2. If $\lim\limits_{x \to -\infty} \frac{u''(x)}{u'(x)} < \infty$, then there exists $\varepsilon > 0$ such that for all $Y = X - \varepsilon'$, $\varepsilon' < \varepsilon$, there exists n^* such that for all $n \geqslant n^*$, $Y^n \succ L^n$.

Consider now the case $\mathrm{E}(X) > 0$. The following proposition demonstrates that if there exists $\tilde{q} \in Q$ for which the expected value of $X_{\tilde{q}}$ is negative, then the implications of Theorem 2, Proposition 1 (of the CEU model) and Proposition 3 (of the smooth model) do not hold.

Proposition 6 Let the MEU preferences satisfy risk aversion. For every ambiguous act L with an anchor lottery X such that $\mathrm{E}(X) > 0$, if there exists $\tilde{q} \in Q$ such that $\mathrm{E}(X_{\tilde{q}}) < 0$ then for a sufficiently large n, $0 \succ L^n$.

6 Discussion

As early as 1961 did William Fellner [8, pp. 678–9] ask: "there is the question whether, if we observe in him [the decision maker] the trait of nonadditivity, he is or is not likely gradually to lose this trait *as he gets used to the uncertainty with which he is faced*." Fellner pointed out a fundamental problem in answering this question empirically: In an experiment, decision makers may understand that the ambiguity is generated by a randomization mechanism and is therefore not ambiguous, but this is not necessarily the case with processes of nature or social life.

Our analysis shows that a lot depends on the way we choose to model ambiguity. But at least under some assumptions, some aspects of ambiguity aversion become insignificant when the decision maker is faced with many similar ambiguous situations within the CEU and the smooth models, and even in the maxmin model. The term "similar" is of course not well defined, but loosely speaking, our analysis shows that even though decision makers don't learn anything new about the world as they face repeated ambiguity, they may still learn not to fear this lack of knowledge.

The proofs of Theorems 1, 3, and 4 reveal another property of preferences as n increases to infinity. Denote by c^n and d^n the certainty equivalents of X^n and L^n. These theorems show that $\lim_{n\to\infty} \frac{d^n}{n} = \lim_{n\to\infty} \frac{c^n}{n}$. This interpretation of the theorems deals with the certainty equivalents per case. An alternative way to analyze attitudes per case is to divide the act L^n and the anchoring lottery X^n by n. The probabilistic lottery will then converge to its average. Maccheroni and Marinacci [18] proved that as $n \to \infty$, the capacity of the event "the average outcome of the ambiguous act L is between its CEU (with the linear utility $u(x) = x$) and minus the CEU value of $-L$" is one. This result however is irrelevant to our analysis, as the difference in the limits of $\frac{L^n}{n}$ and $\frac{X^n}{n}$ does not determine the decision maker's preferences and, in particular, does not imply that $\frac{d^n}{n}$ and $\frac{c^n}{n}$ do not converge to the same limit.

For example, consider EU preferences where $X = (-100, \frac{1}{2}; 200, \frac{1}{2})$, $Y = (1,1)$, and $u(x) = -e^{-ax}$ such that $c^1(X) = 0$. Then, $\lim_{n\to\infty} \frac{X^n}{n} = 50 > 1 \equiv \frac{Y^n}{n}$ while $\frac{c^n(X^n)}{n} = 0 < 1 = \frac{c^n(Y^n)}{n}$. By continuity, a similar example can be created for the CEU model.

Similarly to this extension of the law of large numbers, the central limit theorem of classical probability theory was also extended to the uncertainty framework. This was done by Marinacci [20], who used a certain set of capacities, and by Epstein, Kaido, and Seo [6], who made use of belief functions. The latter authors also studied confidence regions. However, by the preceding argument, and as was argued by Samuelson [25], when decision makers are confronted with sequences like X^n they may not evaluate them by looking at the limit of their average distributions.

Very few experiments checked attitudes to repeated ambiguity (although it seems that several more are currently being conducted). Liu and Colman [17] report that participants chose ambiguous options significantly more frequently in repeated-choice than in single-choice. This suggests that repetition diminishes the effect of ambiguity aversion. Filiz-Ozbay, Gulen, Masatlioglu, and Ozbay [9] report that ambiguity aversion diminishes with the size of the urn. The intuition behind their result agrees with our finding, since both are based on the idea that the more options there are (number of balls to draw from or a larger number of urns) the less plausible is the extreme pessimistic view that Nature always acts against the decision-maker. On the other hand, Halevy and Feltkamp [14] and Epstein and Halevy [5] conducted experiments that involve drawing from two urns and report that when no information regarding the dependence between the urns is provided, individuals display higher ambiguity aversion with respect to it.

Other models imply a connection between CEU and EU. Klibanoff [15] studied the relation between stochastic independence and convexity of the capacity in the CEU model[7] and found that together they imply EU (hence the capacity must be additive). His results are not related to ours since we do not assume stochastic independence and, furthermore, the capacities we analyze are not required to be convex.

[7] Convexity of the capacity ν means that $\nu(E) + \nu(E') \leqslant \nu(E \cup E') + \nu(E \cap E')$. Note however that we do not require the capacities ν^n to be convex. For further analysis of these concepts, see Ghirardato and Marinacci [12] and Chateauneuf and Tallon [2]. See also Machina and Siniscalchi [19].

Appendix A: Proofs

Given the anchor lottery $X^n = (x_1^n, p_1^n; \ldots; x_{k_n}^n; p_{k_n}^n)$, define $g^n : [0,1] \to [0,1]$ such that for $i = 1, \ldots, k_n$,

$$g^n\left(\sum_{j=1}^{i} p_j^n\right) = 1 - \nu^n\left(\bigcup_{j=i+1}^{k_n} E_j^n\right)$$

and let g^n be piecewise linear on the segment $[0, p_{k_n}^n]$ and on the segments $[\sum_{j=1}^{i} p_j^n, \sum_{j=1}^{i+1} p_j^n]$, $i = 1, \ldots, k_n - 1$. Note that by ambiguity aversion for all E, $\nu^n(E) \leqslant P^n(E)$, hence by the piece-wise linearity of g^n, we have $g^n(p) \geqslant p$. Eq. (1) thus becomes

$$\mathrm{CEU}^n(L^n) = u(x_1^n)g^n(p_1^n) + \sum_{i=2}^{k_n} u(x_i^n)\left[g^n\left(\sum_{j=1}^{i} p_j^n\right) - g^n\left(\sum_{j=1}^{i-1} p_j^n\right)\right]$$

Denote by F_Z the distribution of lottery Z. In the sequel we use the integral versions of the expected utility and the CEU models:

$$\mathrm{EU}(X^n) = \int u(z)\, dF_{X^n}(z)$$

$$\mathrm{CEU}^n(L^n) = \int u(z)\, dg^n(F_{X^n}(z))$$

Observe that by the boundedness assumption, for each n, g^n is Lipschitz with K. That is, for all $p > p'$, $g(p) - g(p') \leqslant K(p - p')$. Finally, we denote the certainty equivalents of X^n and L^n by c^n and d^n respectively. In the proofs of this Appendix we show the relationships between $\lim_{n\to\infty} \frac{c^n}{n}$ and $\lim_{n\to\infty} \frac{d^n}{n}$ to obtain the desired results regarding the certainty equivalents of the various repeated lotteries.

The proofs of this appendix use several claims regarding expected utility theory. All these claims are proved as lemmas in Appendix B.

Proof of Theorem 1: We first prove that $\lim_{n\to\infty} \frac{d^n}{n} = \lim_{n\to\infty} \frac{c^n}{n}$. We divide the proof into three cases. Assume throughout, wlg, that $u(0) = 0$ and $u'(0) = 1$. (i) $\lim_{x\to-\infty} u'(x) = H < \infty$: Since u is concave, $\lim_{x\to-\infty} u''(x) = 0$. By Lemma 5 case (i), $\lim_{n\to\infty} \frac{c^n}{n} = \mathrm{E}(X)$. Since for all n, $d^n \leqslant c^n \leqslant n\mathrm{E}(X)$, it is enough to prove that $\lim_{n\to\infty} \frac{d^n}{n} \geqslant \mathrm{E}(X)$. Define $w(x) = \min\{Hx, 0\}$. By assumption, $u(x) \geqslant w(x)$ for all x. Let CEU^n_w denote the CEU^n functional with respect to w. Then $\mathrm{CEU}^n(L^n) \geqslant \mathrm{CEU}^n_w(L^n)$. Hence for sufficiently large n

$$\begin{aligned} u(d^n) \geqslant \mathrm{CEU}^n_w(L^n) &= \int w(z)\, dg^n(F_{X^n}(z)) = H \int_{z\leqslant 0} z\, dg^n(F_{X^n}(z)) \\ &\geqslant KH \int_{z\leqslant 0} z\, dF_{X^n}(z) \geqslant KH \left(\frac{x_1\sigma^2}{n^{2(\alpha-1)}} - n^\alpha + n\mathrm{E}(X) \right) \end{aligned}$$

where the last inequality follows by Lemma 2. Since u is concave and $u'(0) = 1$, $d^n \geqslant KH(\frac{x_1\sigma^2}{n^{2(\alpha-1)}} - n^\alpha + n\mathrm{E}(X))$. Therefore, $\lim_{n\to\infty} \frac{d^n}{n} \geqslant KH \lim_{n\to\infty} (\frac{x_1\sigma^2}{n^{2\alpha-1}} - \frac{1}{n^{1-\alpha}} + \mathrm{E}(X)) = \mathrm{E}(X) = 0$.

(ii) $\lim_{x\to-\infty} u'(x) = \infty$: We have

$$\begin{aligned} \lim_{n\to\infty} \frac{\mathrm{CEU}^n(L^n)}{\mathrm{EU}(X^n)} &\leqslant \lim_{n\to\infty} \frac{\int_{x<0} u(x)\, dg^n(F_{X^n}(x))}{\mathrm{EU}(X^n)} \\ &\leqslant \lim_{n\to\infty} \frac{K \int_{x<0} u(x)\, dF_{X^n}(x)}{\mathrm{EU}(X^n)} = K \end{aligned}$$

The second inequality follows by the fact that all the g^n functions are Lipschitz with the same value of K and the equality is obtained by Lemma 3. Observe that numerators and denominators of all expressions are negative. It thus follows that for sufficiently large n,

$$u(d^n) \geqslant (K+1)u(c^n) \tag{3}$$

Let $a = \lim_{x\to-\infty} -\frac{u''(x)}{u'(x)}$. We consider two cases:

(ii-a) $a = 0$: Since u is concave, $u(0) = 0$, and $u'(0) = 1$, $(K+1)u(c^n) \geqslant u((K+1)c^n)$, implying $d^n \geqslant (K+1)c^n$. By Lemma 5 case (ii), $\lim_{n\to\infty} \frac{c^n}{n} = 0$, hence $\lim_{n\to\infty} \frac{d^n}{n} = 0$.

(ii-b) $a > 0$: It follows by the concavity of u and by the fact that $d^n \leqslant c^n$ that

$$\frac{u(c^n) - u(d^n)}{c^n - d^n} \geqslant u'(c^n)$$

hence by inequality (3), for sufficiently large n,

$$c^n - d^n \leqslant \frac{u(c^n) - u(d^n)}{u'(c^n)} \leqslant -\frac{Ku(c^n)}{u'(c^n)}$$

By l'Hopital's rule, since $\lim\limits_{x\to-\infty} u(x) = -\infty$ and $\lim\limits_{x\to-\infty} u'(x) = \infty$, $\lim\limits_{x\to-\infty} -\frac{u'(x)}{u(x)} = \lim\limits_{x\to-\infty} -\frac{u''(x)}{u'(x)} = a > 0$. By Lemma 4, $\lim\limits_{n\to\infty} c^n = -\infty$, hence for a sufficiently large n,

$$-\frac{Ku(c^n)}{u'(c^n)} \leqslant \frac{K+1}{a} \implies 0 \leqslant \frac{c^n}{n} - \frac{d^n}{n} \leqslant \frac{K+1}{an} \underset{n\to\infty}{\longrightarrow} 0$$

It thus follows that $\lim\limits_{n\to\infty} \frac{d^n}{n} = \lim\limits_{n\to\infty} \frac{c^n}{n}$.

Denote this common limit $\hat{c}$. By Lemma 5, $\hat{c}$ is the certainty equivalent of X under v, where $v(x) = x$ if $a = 0$, and $v(x) = -e^{-ax}$ if $a > 0$. Consider Y dominated by X by strict FOSD, and let $\hat{b} < \hat{c}$ be the certainty equivalent of Y under v. Let b^n be the certainty equivalent of Y^n under u. By Lemma 5, $\lim\limits_{n\to\infty} \frac{b^n}{n} = \hat{b}$, hence $\lim\limits_{n\to\infty} \frac{b^n}{n} < \lim\limits_{n\to\infty} \frac{d^n}{n}$. It thus follows that for sufficiently large n, $d^n > b^n$, hence $L^n \succ Y^n$. ■

Proof of Theorem 2: Assume wlg that $u(0) = 0$, $u'(0) = 1$, that $n_0 = 1$, and hence $c^n \geqslant 0$ for all n. Assume first that $\lim\limits_{x\to-\infty} u'(x) = \infty$. Define $u^n(x) = u(x) - u(nx_m)$ and note that $u^n(nx_m) = 0$ and $u^n(x) < 0$, for all outcomes of X^n. These inequalities and the boundedness assumption imply that for the $\text{CEU}^n_{u^n}$, the CEU^n functional with respect to u^n,

$$\begin{aligned} u^n(d^n) = \text{CEU}^n_{u^n}(L^n) &= \int u^n(z)\, dg^n(F_{X^n}(z)) \\ &\geqslant K \int u^n(z)\, dF_{X^n}(z) \geqslant Ku^n(c^n) \end{aligned}$$

The inequality $u^n(c^n) \geqslant u^n(0)$ yields $u^n(d^n) \geqslant Ku^n(0)$.

Going back to u, noting that $1-K \leqslant 0$ and that, by concavity, $u(nx_m) \leqslant nu(x_m)$,

$$\begin{aligned} u(d^n) &= u^n(d^n) + u(nx_m) \geqslant Ku^n(0) + u(nx_m) \\ &= -Ku(nx_m) + u(nx_m) = (1-K)u(nx_m) \\ &\geqslant n(1-K)u(x_m) \end{aligned}$$

Denote $A = (1-K)u(x_m)$. By assumption, $A \leqslant 0$. Note that the concavity of u and $\lim\limits_{x\to-\infty} u'(x) = \infty$ imply $\lim\limits_{y\to-\infty} u^{-1}(y)/y = 0$. Then, $d^n \geqslant u^{-1}(nA)$ implies $\lim\limits_{n\to\infty} \frac{d^n}{n} \geqslant \lim\limits_{n\to\infty} \left(\frac{u^{-1}(nA)}{nA}\right) A = 0$.

Finally, if $\lim\limits_{x\to-\infty} u'(x) = H < \infty$, then proceed as in case (i) in the proof of Theorem 1. Note that since $\mathrm{E}(X) > 0$, Lemma 2 implies $\int_{x\leqslant 0} x\, dF_{X^n}(x) \geqslant \frac{x_1\sigma^2}{n^{2(\alpha-1)}} - n^\alpha$ for sufficiently large n.

Similarly to the last paragraph in the proof of Theorem 1, replacing X with 0 implies that for a sufficiently large n, $L^n \succ Y^n$. ■

Proof of Proposition 1: Assume wlg that $u(x) < 0$ for all x and that $\lim\limits_{x\to\infty} u(x) = 0$. Then

$$u(d^n) = \mathrm{CEU}^n(L^n) = \int u(z)\, dg^n(F_{X^n}(z)) \geqslant K \int u(z)\, d(F_{X^n}(z)) \geqslant Ku(c^n)$$

Since $X^n \succ n\varepsilon$ for a sufficiently large n, we have $c^n > n\varepsilon$. As $n\varepsilon$ goes to infinity, $\lim\limits_{n\to\infty} u(c^n) = 0$ and, by the above argument, $\lim\limits_{n\to\infty} u(d^n) = 0$. This implies the existence of n^* such that for all $n > n^*$, $u(d^n) > u(0)$. For these n, $d^n > 0$ and $L^n \succ 0$. ■

Proof of Proposition 2: The case of linear u with zero expected value is covered by case (i) in the proof of Theorem 1 (note that, by construction, $c^n = 0$ for all n). Assume that the expected value is not zero. It follows from eq. (1) that since the utility is linear, $\mathrm{CEU}^n(\tilde{L}+\eta) = \mathrm{CEU}^n(\tilde{L}) + \eta$ for all $\tilde{L}$. Denote $\hat{X} = X - \mathrm{E}(X)$ and $\hat{L} = L - \mathrm{E}(X)$, and let $\hat{d}^n \sim \hat{L}^n$. By the above, $\mathrm{E}(\hat{L}) = 0$ implies $\lim\limits_{n\to\infty} \frac{\hat{d}^n}{n} = 0$. hence

$$\begin{aligned} d^n &= CEU^n(L^n) = CEU^n((\hat{L} + \mathrm{E}(X))^n) \\ &= CEU^n(\hat{L}^n + n\mathrm{E}(X)) = CEU^n(\hat{L}^n) + n\mathrm{E}(X) = \hat{d}^n + n\mathrm{E}(X) \end{aligned}$$

implies $\lim\limits_{n\to\infty} \frac{d^n}{n} = \mathrm{E}(X)$. Clearly, $\frac{c^n}{n} = \frac{n\mathrm{E}(X)}{n} = \mathrm{E}(X)$ as well.

Let $u(x) = -e^{-ax}$ with $a > 0$. By Lemma 1, $c^n = nc^1$ and hence $\lim\limits_{n\to\infty} \frac{c^n}{n} = c^1$. By the definitions of c^1 and d^n we have

$$\begin{aligned} \mathrm{EU}(X - c^1) &= \int -e^{-az}\, dF_{X-c^1}(z) = \int -e^{-a(z-c^1)}\, dF_X(z) \\ &= e^{ac^1} \int -e^{-az}\, dF_X(z) = e^{ac^1}(-e^{-ac^1}) = -1 \end{aligned} \tag{4}$$

and

$$\begin{aligned}
&\mathrm{CEU}^n\left(\left(L-\tfrac{d^n}{n}\right)^n\right)=\int -e^{-az}\,dg^n\left(F_{\left(X-\frac{d^n}{n}\right)^n}(z)\right)= \\
&\int -e^{-az}\,dg^n(F_{X^n-d^n}(z))=\int -e^{-a(z-d^n)}\,dg^n\left(F_{X^n}(z)\right)= \\
&e^{ad^n}\int -e^{-az}\,dg^n\left(F_{X^n}(z)\right)=e^{ad^n}\left(-e^{-ad^n}\right)=-1
\end{aligned} \tag{5}$$

The sequence $\{\frac{d^n}{n}\}_{n=1}^{\infty}$ is bounded (since the support of X is) and assume, by way of negation, that the sequence does not converge to c^1. Then, wlg there exists $\varepsilon>0$ and a subsequence $\{\frac{d^{n_j}}{n_j}\}_{j=1}^{\infty}$ satisfying $\lim_{j\to\infty}\frac{d^{n_j}}{n_j}<c^1-\varepsilon$. Without loss of generality, assume that for all j, $\frac{d^{n_j}}{n_j}<c^1-\varepsilon$. Hence,

$$\begin{aligned}
&\mathrm{CEU}^n\left(\left(L-\tfrac{d^{n_j}}{n_j}\right)^{n_j}\right)=\int -e^{-az}\,dg^n\left(F_{(X-d^{n_j}/n_j)^{n_j}}(z)\right) \\
> &\int -e^{-az}\,dg^n\left(F_{(X-c^1+\varepsilon)^{n_j}}\right)(z)\geqslant -K\int e^{-az}\,dF_{(X-c^1+\varepsilon)^{n_j}}(z) \\
= &-K\left[\int e^{-az}\,dF_{X-c^1+\varepsilon}(z)\right]^{n_j}=-Ke^{-an_j\varepsilon}\left[\int e^{-az}\,dF_{X-c^1}(z)\right]^{n_j} \\
= &-Ke^{-an_j\varepsilon}\underset{j\to\infty}{\longrightarrow}0
\end{aligned}$$

where the last equality follows by eq. (4). Therefore, for sufficiently large j,

$$\mathrm{CEU}^n\left(\left(L-\tfrac{d^{n_j}}{n_j}\right)^{n_j}\right)>-1$$

in contradiction with eq. (5). To conclude, here too $\lim_{n\to\infty}\frac{d^n}{n}=c^1=\lim_{n\to\infty}\frac{c^n}{n}$.

We now continue as in the last paragraph in the proof of Theorem 1, noting that here $u=v$, hence $Y\prec X$ implies $\hat{b}<\hat{c}$. ■

Example 2 Boundedness is not necessary for our results. Let $u(x)=-e^{-x}$ and let $\nu^n(E)=1-\sqrt{1-P^n(E)}$. These capacities do not satisfy the boundedness assumption. To see why, let $E^{n\prime}=\{(G,\dots,G)\}$ and let $E^n=\neg E^{n\prime}$. We obtain

$$\nu^n(E^n\cup E^{n\prime})-\nu^n(E^n)=1-\left(1-\sqrt{1-\frac{2^n-1}{2^n}}\right)=\frac{1}{\sqrt{2^n}}$$

The ratio between this difference and 2^{-n}, the probability of $E^{n\prime}$, is $\sqrt{2^n}$, which is not bounded by any K.

For Theorem 1, consider the ambiguous act $L = (-0.5, E_1; 0.5,\ E_2)$ with the anchor lottery $X = (-0.5, \frac{1}{2}; 0.5, \frac{1}{2})$. Let $Y = (-0.55, \frac{1}{2}; 0.45, \frac{1}{2})$. The certainty equivalent of Y^n is $-0.17n$ and that of L^n is $-0.21n$.

For the other results, consider the act $L = (-.35, E_1;\ 0.65, E_2)$ with the anchor lottery $X = (-0.35, \frac{1}{2}; 0.65, \frac{1}{2})$ and let $Y = (-0.02, 1)$. The certainty equivalent of X^n is $0.03n$, while that of Y^n is $-0.02n > -0.06n$, which is larger than the certainty equivalent of L^n.

Example 3 The boundedness of u from above is required for Proposition 1. Let $X = (-\frac{1}{4}, \frac{1}{2};\ \frac{3}{4}, \frac{1}{2})$. Define ν^n as in example 1. We get

$$\mathrm{EU}(X^{4n}) = \sum_{i=-n}^{3n} \binom{4n}{i+n} \frac{1}{2^{4n}} u(i) \tag{6}$$

$$\mathrm{CEU}^n(L^{4n}) = 2\sum_{i=-n}^{n-1} \binom{4n}{i+n} \frac{1}{2^{4n}} u(i) + \binom{4n}{2n} \frac{1}{2^{4n}} u(n) \tag{7}$$

Let $u(x) = x$ for $x \geqslant 0$. We define $u(-n)$ inductively. Let

$$\begin{aligned} v_n &= -\sum_{i=-n+1}^{-1} \binom{4n}{i+n} u(i) - \sum_{i=1}^{n-1} \binom{4n}{i+n} i - \binom{4n}{2n} \frac{n}{2} \qquad (8) \\ w_n &= 2u(-n+1) - u(-n+2) \end{aligned}$$

and define u for $x < 0$ as follows. For $n = 1, \ldots,$ let $u(-n) = \min\{v_n, w_n\}$, and for $x \in (-n, -n+1)$ let $u(x) = u(-n) + (x+n)[u(-n+1) - u(-n)]$. The function u is strictly increasing and weakly concave.

Claim 1 $\lim_{n\to\infty} u(-n)/n = -\infty$.

Proof: Suppose not. Then there exists $A > 0$ such that for all n, $-u(-n)/n \leqslant A$, and since between $-n$ and $-n+1$ the function u is linear, it follows that for all n, $-u(-n)/n \leqslant A$.

By definition, $u(-n) \leqslant v_n$, hence it follows by eqs. (7) and (8) that for all n, $\mathrm{CEU}^n(X^{4n}) \leqslant 0$. On the other hand, by eq. (7),

$$\begin{aligned}
\mathrm{CEU}^n(X^{4n}) &= 2\sum_{i=-n}^{-1}\binom{4n}{i+n}\frac{u(i)}{2^{4n}} + 2\sum_{i=1}^{n-1}\binom{4n}{i+n}\frac{i}{2^{4n}} + \binom{4n}{2n}\frac{n}{2^{4n}} \\
&\geqslant -\frac{(n-1)nA}{2^{4n-1}}\binom{4n}{n-1} + 1 \times \left[\frac{1}{2} - \Pr(X^{4n} \leqslant 0)\right] \qquad (9)
\end{aligned}$$

Let $\beta_n = \frac{(n-1)nA}{2^{4n-1}}\binom{4n}{n-1}$. Clearly

$$\begin{aligned}
\frac{\beta_{n+1}}{\beta_n} &= \frac{n(n+1)A2^{4n-1}\binom{4n+4}{n}}{(n-1)nA2^{4n+3}\binom{4n}{n-1}} \\
&= \frac{(n+1)(4n+4)(4n+3)(4n+2)(4n+1)}{16(n-1)n(3n+4)(3n+3)(3n+2)} \to \frac{4^4}{16 \times 3^3} = \frac{16}{27}
\end{aligned}$$

Hence $\lim\limits_{n\to\infty}\beta_n = 0$. Likewise, $\Pr(X^{4n} \leqslant 0) \leqslant \frac{n}{2^{4n}}\binom{4n}{n} \to 0$, hence the expression of eq. (9) converges to $\frac{1}{2}$; a contradiction. $\square$

Define $n_0 = 0$, and let n_i satisfy

1. $u(-n_i) = v_{n_i}$
2. For $n_{i-1} < j < n_i$, $u(-j) < v_j$

It follows by Claim 1 that $\{n_i\}$ is not a finite sequence, as otherwise the function u would become linear from a certain point on to the left and will never intersect the line Ax for sufficiently high A.

By definition, $\mathrm{CEU}^n(X^{4n_i}) = 0$ and $d^{4n_i} = 0$. It thus follows by eq. (6) that

$$\begin{aligned}
u(c^{4n_i}) = \mathrm{EU}(X^{4n_i}) &= \left[\binom{4n_i}{2n_i}\frac{n_i}{2} + \sum_{i=n_i+1}^{3n_i}\binom{4n_i}{i+n_i}i\right]\frac{1}{2^{4n_i}} \\
&> \frac{n_i}{2} \times \Pr\left(X^{4n_i} \geqslant n_i\right) = \frac{n_i}{4}
\end{aligned}$$

Since it is positive, $u(c^{4n_i}) = c^{4n_i}$, hence $\lim\limits_{i\to\infty}\frac{c^{4n_i}}{4n_i} \geqslant \frac{1}{16}$ while $\frac{d^{4n_i}}{4n_i} \equiv 0$. $\blacksquare$

Proof of Theorem 3: The certainty equivalents are defined by $u(c^n) = \mathrm{EU}^u(X^n)$ and $\phi(d^n) = \mathrm{SM}^{\phi u}(L^n)$.[8] By ambiguity aversion, ϕ is more concave than u, hence $\mathrm{SM}^{\phi\phi}(L^n) \leqslant \mathrm{SM}^{\phi u}(L^n) \leqslant \mathrm{SM}^{uu}(L^n)$. Let $\bar{d}^n$ be the certainty equivalent of L^n under $\mathrm{SM}^{\phi\phi}$ and note that c^n is the certainty equivalent of SM^{uu} (since $\mathrm{SM}^{uu}(L^n) = \mathrm{EU}^u(X^n)$). Hence $\bar{d}^n \leqslant d^n \leqslant c^n$ for all n and

$$\lim_{n\to\infty} \frac{\bar{d}^n}{n} \leqslant \lim_{n\to\infty} \frac{d^n}{n} \leqslant \lim_{n\to\infty} \frac{c^n}{n}$$

Using $\mathrm{SM}^{\phi\phi}(L^n) = \mathrm{EU}^\phi(X^n)$, Lemma 5 implies $\lim\limits_{n\to\infty} \frac{\bar{d}^n}{n} = \lim\limits_{n\to\infty} \frac{c^n}{n}$. Hence, $\lim\limits_{n\to\infty} \frac{d^n}{n} = \lim\limits_{n\to\infty} \frac{c^n}{n}$. The rest of the proof is similar to the last paragraph in the proof of Theorem 1. ■

Proof of Proposition 3: Ambiguity aversion implies that ϕ is more concave than u, hence $\mathrm{SM}^{\phi u}(L^n) \geqslant \mathrm{SM}^{\phi\phi}(L^n) = \mathrm{EU}^\phi(X^n)$.

Let $v_t(x) = -e^{-tx}$. Suppose first that u is linear. By assumption $\mathrm{E}(X) > 0$ and $\lim\limits_{x\to-\infty} \frac{\phi''(x)}{\phi'(x)} = 0$. By Lemma 6, $\lim\limits_{x\to-\infty} \frac{\phi''(x)}{\phi'(x)} = 0$ implies that for every t there exists y such that $\phi(x) > v_t(x)$ for all $x < y$. Hence (see Nielsen [22, Prop. 1]) for sufficiently large n, $\mathrm{EU}^\phi(X^n) > \phi(0)$, implying that for all such n, $L^n \succ 0$.

Next suppose that $u(x) = -e^{-sx}$ for some s. Then $\lim\limits_{x\to-\infty} -\frac{u''(x)}{u'(x)} = \lim\limits_{x\to-\infty} -\frac{\phi''(x)}{\phi'(x)} = s$ and, by assumption, $\mathrm{EU}^u(X^n) > u(0) = -1$. By continuity, for t close to s, $\mathrm{EU}^{v_t}(X) > v_t(0) = -1$. Choose such $t > s$ and note (Lemmas 6) that there exists y such that $\phi(x) > v_t(x)$ for all $x < y$. Wlg assume $y < 0$ and $\phi(0) = 0$. Then

$$\begin{aligned}
\int_{x<0} \phi(x)\, dF_{X^n}(x) &= \int_{x<y} \phi(x)\, dF_{X^n}(x) + \int_y^0 \phi(x)\, dF_{X^n}(x) \\
&\geqslant \int_{x<y} v_t(x)\, dF_{X^n}(x) + \phi(y)\Pr(y \leqslant X^n < 0) \\
&\geqslant \mathrm{EU}^{v_t}(X^n) + \phi(y)\Pr(y \leqslant X^n < 0) \\
&= -(\mathrm{EU}^{v_t}(X))^n + \phi(y)\Pr(y \leqslant X^n < 0) \underset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

[8] The certainty equivalent of the smooth model is computed using ϕ since $\mathrm{SM}^{\phi u}(x, s_1; \ldots; x, s_n) = \phi(x)$.

where the limit is 0 because $\mathrm{EU}^{v_t}(X) \in (-1, 0)$ and $\lim_{n\to\infty} \Pr(y \leqslant X^n < 0) = 0$. As $\lim_{n\to\infty} \mathrm{EU}^{\phi}(X^n)_{x \geqslant 0} = \sup_x \phi(x)$, we conclude that for sufficiently large n, $\mathrm{EU}^{\phi}(X^n) > \phi(0)$ and $L^n \succ 0$. ■

Proof of Proposition 4: If the risk aversion of ϕ is bounded from below by t and u is concave, then for every n, $d_u^n \leqslant \bar{d}^n$, where d_u^n is the certainty equivalent of L^n under u and ϕ and $\bar{d}^n$ is the certainty equivalent of L^n under the functions $\bar{u}(x) = x$ and $\phi^*(x) = -e^{-tx}$.

Denote $z_i = \mathrm{E}(X_{p^i})$, $Z = (z_1, \mu^1; \ldots; z_\ell, \mu^\ell)$ and note that

$$\mathrm{E}(Z) = \sum_{i=1}^{\ell} \mu^i \mathrm{E}(X_{p^i}) = \mathrm{E}\left(\textstyle\sum_{i=1}^{\ell} \mu^i X_{p^i}\right) = \mathrm{E}(X) = 0$$

If the decision maker is using ϕ^* and $\bar{u}$, then

$$\begin{aligned} \mathrm{SM}^{\phi^*\bar{u}}(L) &= \sum_{i=1}^{\ell} \mu^i \cdot \phi^* \circ \bar{u}^{-1}(\mathrm{EU}^{\bar{u}}(X_{p^i})) = \sum_{i=1}^{\ell} \mu^i \phi^*(\mathrm{E}(X_{p^i})) \\ &= \sum_{i=1}^{\ell} \mu^i \phi^*(z_i) = \mathrm{EU}^{\phi^*}(Z) \end{aligned}$$

Also, it follows from eq. (2) that

$$\mathrm{SM}^{\phi^*\bar{u}}(L^n) = \sum_{j=1}^{\ell^n} \mu_j^n \cdot \phi^* \circ \bar{u}^{-1}(\mathrm{EU}^{\bar{u}}(Y_j^n)) = \sum_{j=1}^{\ell^n} \mu_j^n \phi^*[\mathrm{E}(Y_j^n)]$$

The expected value of Y_j^n is the sum of the expected values of the sequence of lotteries it represents. As there are in this sequence j_i lotteries of type X_{p^i}, $i = 1, \ldots, \ell$, the expected value of Y_j^n is $\sum_{i=1}^{\ell} j_i \mathrm{E}(X_{p^i})$. Hence

$$\begin{aligned} \sum_{j=1}^{\ell^n} \mu_j^n \phi^*[\mathrm{E}(Y_j^n)] &= \sum_{j=1}^{\ell^n} \mu_j^n \phi^* \left[\left(\textstyle\sum_{i=1}^{\ell} j_i \mathrm{E}(X_{p^i})\right)\right] \\ &= \sum_{j=1}^{\ell^n} \mu_j^n \phi^* \left[\left(\textstyle\sum_{i=1}^{\ell} j_i z_i\right)\right] = \mathrm{EU}^{\phi^*}(Z^n) \end{aligned}$$

Where the last equation follows by the fact that $\sum_{i=1}^{\ell} j_i z_i$ is an outcome of the lottery Z^n which is obtained from playing n times lottery Z. We obtain that

$$\bar{d}^n = (\phi^*)^{-1}(\mathrm{SM}^{\phi^* \bar{u}}(L^n)) = (\phi^*)^{-1}(\mathrm{EU}^{\phi^*}(Z^n))$$

And since ϕ^* is exponential, Lemma 1 implies $\frac{\bar{d}^n}{n} = \bar{d}^1 = (\phi^*)^{-1}(\mathrm{EU}^{\phi^*}(Z)) < 0$.

Consider the utility function $v^s(x) = -e^{-sx}$. Since this function represents constant absolute risk aversion, it follows that for this function, the average certainty equivalent of X^n, $\frac{\bar{c}^n}{n}$, equals the certainty equivalent of X, $\bar{c}^1$. As in the proof of Lemma 5 case (ii), as $s \to 0$, $\bar{c}^1 \to 0$ as well.

If u is less risk averse than v^s, then $\lim\limits_{n\to\infty} \frac{c_u^n}{n}$ computed with respect to u will be at least as high as that of v^s. By the first part of the proof $\lim\limits_{n\to\infty} \frac{d_u^n}{n} \leqslant \bar{d}^1 < 0$. For sufficiently small s we can get $\bar{c}^1$ as close as we wish to zero, and in particular, for small s, $\lim\limits_{n\to\infty} \frac{d_u^n}{n} \leqslant \lim\limits_{n\to\infty} \frac{\bar{d}^n}{n} = \bar{d}^1 < \bar{c}^1 = \lim\limits_{n\to\infty} \frac{\bar{c}^n}{n} \leqslant \lim\limits_{n\to\infty} \frac{c_u^n}{n}$. Let $s^* > 0$ be such a value of s and denote $v^* = v^{s^*}$.

Let $\mathcal{N} = \{Y : \mathrm{EU}^{v^*}(Y) > v^*(\bar{d}^1)\}$. Clearly, X is in the interior of $\mathcal{N}$. Let $Y \in \mathcal{N}$. Let b_u^n be the certainty equivalent of Y^n under the utility function u. Since v^* is exponential, $\lim\limits_{n\to\infty} \frac{b_{v^*}^n}{n} = v^{*-1}(\mathrm{EU}^{v^*}(Y)) > \bar{d}^1$.

Choose now u which is less risk averse than v^*. Then for every n, $b_u^n \geqslant b_{v^*}^n$, hence $\lim\limits_{n\to\infty} \frac{b_u^n}{n} \geqslant \lim\limits_{n\to\infty} \frac{b_{v^*}^n}{n} > \bar{d}^1 \geqslant \lim\limits_{n\to\infty} \frac{d_u^n}{n}$. Therefore for every such u there is a sufficiently large n such that under this u, $Y^n \succ L^n$. ■

Proof of Proposition 5: By construction,

$$u(c^1) = \mathrm{EU}^u(X) = \mathrm{EU}^u\left(\sum_{i=1}^{\ell} \mu^i X_{p^i}\right) = \sum_{i=1}^{\ell} \mu^i \mathrm{EU}^u(X_{p^i})$$

and

$$(\phi \circ u^{-1})(u(d^1)) = \phi(d^1) = \sum_{i=1}^{\ell} \mu^i (\phi \circ u^{-1})(\mathrm{EU}^u(X_{p^i}))$$

Rewriting the equations and denoting $h = \phi \circ u^{-1}$ yields

$$u(c^1) = \sum_{i=1}^{\ell} \mu^i \mathrm{EU}^u(X_{p^i}) \text{ and } h(u(d^1)) = \sum_{i=1}^{\ell} \mu^i h(\mathrm{EU}^u(X_{p^i}))$$

Assume first that ϕ is exponential of the form $\phi(x) = -e^{-tx}$. If u is linear, then the proof of the first part of Proposition 4 implies $\frac{d^n}{n} = d^1 < 0 = \frac{c^n}{n}$. Next, consider exponential $u(x) = -e^{-sx}$ where, by assumption, $s > 0$. Since $t > s$, $h(y) = -(-y)^{t/s}$ is strictly concave and increasing. Then, the above equations imply $u(d^1) < u(c^1)$ and $d^1 < c^1$.

By Lemma 1, $\frac{c^n}{n} = c^1$ for all n and hence $\lim\limits_{n\to\infty} \frac{c^n}{n} = c^1$. Moreover, denoting $c_i = u^{-1}(\mathrm{EU}^u(X_{p^i}))$ and using Lemma 1, for any sequence of lotteries $Y_u^n = (X_{p^1})^{n^1}, \ldots, (X_{p^\ell})^{n^\ell}$, $n^i \in \{0, \mathbb{N}\}$,

$$\mathrm{EU}^u((X_{p^1})^{n^1} \ldots (X_{p^\ell})^{n^\ell}) = -|\,\mathrm{EU}^u(X_{p^1})\,|^{n^1} \times \ldots \times |\,\mathrm{EU}^u(X_{p^\ell})\,|^{n^\ell} =$$
$$-\left(e^{-sc_1}\right)^{n^1} \times \ldots \times \left(e^{-sc_\ell}\right)^{n^\ell} = -e^{-s(n^1c_1+\ldots+n^\ell c_\ell)} = u(n^1c_1 + \ldots + n^\ell c_\ell)$$

Therefore, denoting $C = (c_1, \mu^1; ...; c_\ell, \mu^\ell)$, $\mathrm{SM}^{\phi u}(L^n)$ can be written as $\mathrm{EU}^\phi(C^n)$ for all n:

$$\mathrm{SM}^{\phi u}(L) = \sum_{i=1}^{\ell} \mu^i \phi[u^{-1}(\mathrm{EU}^u(X_{p^i}))] = \sum_{i=1}^{\ell} \mu^i \phi(c_i) = \mathrm{EU}^\phi(C)$$

and

$$\mathrm{SM}^{\phi u}(L^n) = \sum_{j=1}^{(\ell)^n} \mu_j^n \phi[u^{-1}(\mathrm{EU}^u(Y_j^n))] = \sum_{j=1}^{(\ell)^n} \mu_j^n \phi[n^1c_1 + \ldots + n^\ell c_\ell] = \mathrm{EU}^\phi(C^n)$$

Using $d^1 = \phi^{-1}(\mathrm{EU}^\phi(C))$ and $d^n = \phi^{-1}(\mathrm{EU}^\phi(C^n))$, Lemma 1 implies $\frac{d^n}{n} = d^1$ for all n and hence $\lim\limits_{n\to\infty} \frac{d^n}{n} = d^1$ and $\lim\limits_{n\to\infty} \frac{d^n}{n} < \lim\limits_{n\to\infty} \frac{c^n}{n}$.

Finally, if ϕ is not exponential, then repeat the above analysis for the less concave function $\bar{\phi}(x) = -e^{-tx}$ where $t > s$. The last inequality then follows from the fact that for all n, the certainty equivalent of L^n under $\mathrm{SM}^{\phi u}$ is (weakly) smaller than that under $\mathrm{SM}^{\bar{\phi} u}$. The claim of the proposition follows similarly to the last two paragraphs of the proof of Proposition 4. ■

Proof of Theorem 4: Consider first the case $\lim\limits_{x\to-\infty} \frac{u''(x)}{u'(x)} = \infty$. Let $\hat{c}(s)$ be the certainty equivalent of X under the utility function $-e^{-sx}$, for all $s > 0$. Since there exists M such that on $(-\infty, M)$, u is more concave than $-e^{-sx}$, and since $\lim\limits_{n\to\infty} \frac{c^n}{n}$ does not depend on the values of u on $[M, \infty)$ (Conclusion 1), we have $\lim\limits_{n\to\infty} \frac{c^n}{n} \leqslant \hat{c}(s)$ for all s. Next we show that $\lim\limits_{s\to\infty} \hat{c}(s) = x_1$

(note that $\hat{c}(s) = -\frac{1}{s}\ln\left(\sum p_i e^{-sx_i}\right)$). Using l'Hopital's rule we get

$$\lim_{s\to\infty}\hat{c}(s) = \lim_{s\to\infty}\frac{\sum p_i x_i e^{-sx_i}}{\sum p_i e^{-sx_i}} = \lim_{s\to\infty}\frac{p_1x_1 + \sum_{i>1} p_i x_i e^{-s(x_i-x_1)}}{p_1 + \sum_{i>1} p_i e^{-s(x_i-x_1)}} = x_1$$

which, noting that $c^n \geqslant nx_1$ and hence $\frac{c^n}{n} \geqslant x_1$, implies $\lim\limits_{n\to\infty}\frac{c^n}{n} = x_1$. Similarly, for $Y = X-\varepsilon$, the certainty equivalent b^n of Y^n satisfies $\lim\limits_{n\to\infty}\frac{b^n}{n} = x_1-\varepsilon$. Now $d^n \geqslant nx_1$ implies $\lim\limits_{n\to\infty}\frac{b^n}{n} = x_1-\varepsilon < x_1 \leqslant \lim\limits_{n\to\infty}\frac{d^n}{n}$, hence for a sufficiently large n, $L^n \succ Y^n$.

Next, consider the case $\lim\limits_{x\to-\infty}\frac{u''(x)}{u'(x)} = a \in (0,\infty)$. By Lemma 5 case (iii), $\lim\limits_{n\to\infty}\frac{c^n}{n} = \hat{c}$ where $\hat{c}$ is the certainty equivalent of X under the utility $v(x) = -e^{-ax}$. Let $\hat{q} \in Q$ be a probability vector such that X strictly FOSD dominates $X_{\hat{q}}$ and let $\hat{d}$ denote the certainty equivalent of $X_{\hat{q}}$ under v. Clearly, $\hat{d} < \hat{c}$. Define $\hat{d}^n = u^{-1}(\mathrm{EU}\left(X^n_{\hat{q}}\right))$ and observe that, by Lemma 5 case (iii), $\lim\limits_{n\to\infty}\frac{\hat{d}^n}{n} = \hat{d}$. Since, by construction, $d^n \leqslant \hat{d}^n$, we get $\lim\limits_{n\to\infty}\frac{d^n}{n} \leqslant \hat{d}$. Now let ε such that the certainty equivalent of $X-\varepsilon$ under v is $\hat{d}$. Let $Y = X-\varepsilon'$ where $\varepsilon' < \varepsilon$. Again by Lemma 5 case (iii), b^n, the certainty equivalent of Y^n, satisfies $\lim\limits_{n\to\infty}\frac{b^n}{n} > \hat{d}$. Therefore, for a sufficiently large n, $Y^n \succ L^n$.

The case $\lim\limits_{x\to-\infty}\frac{u''(x)}{u'(x)} = 0$ is similarly proved (by replacing the exponential function v with a linear function). ■

Proof of Proposition 6: By definition, $X^n_{\tilde{q}} \succeq L^n$. As $\mathrm{E}(X^n_{\tilde{q}}) < 0$, it follows by risk aversion that $0 \succ X^n_{\tilde{q}} \succeq L^n$. ■

Appendix B: Expected Utility

Except for in Lemmas 1 and 2, which apply to all lotteries X, we assume throughout that $\mathrm{E}(X) \leqslant 0$. In Lemmas 3–6 we assume wlg that the value of all utility functions is zero at zero and that their derivative there is 1.

Lemma 1 Let $u(x) = -e^{-ax}$. Then for lotteries $X_1,\ldots,X_k$, $\mathrm{EU}(\sum_{i=1}^k X_i) = u(\sum_{i=1}^k \mathrm{CE}(X_i))$, where $\mathrm{CE}(X)$ is the certainty equivalent of X. In particular, if $X_i = X$ for all i, then for all n, $\frac{c^n}{n} = c^1$.

Proof: The proof follows from a property of the moment generating functions (see Bulmer [1]). ■

Lemma 2 There exists n_0 such that for all $n > n_0$, $\int_{z\leqslant 0} z\, dF_{X^n}(z) \geqslant \frac{x_1\sigma^2}{n^{2(\alpha-1)}} - n^\alpha + n\mathrm{E}(X)$

Proof: As σ^2 be the variance of X, $n\sigma^2$ is the variance of X^n. Choose $\frac{1}{2} < \alpha < 1$. By Chebyshev's inequality,

$$\Pr(X^n < n\mathrm{E}(X) - n^\alpha) \leqslant \frac{n\sigma^2}{n^{2\alpha}} = \frac{\sigma^2}{n^{2\alpha-1}}$$

There exists n_0 sufficiently large such that for all $n > n_0$, $nx_1 < n\mathrm{E}(X) - n^\alpha$. Then

$$\begin{aligned}\int_{z\leqslant 0} z\, dF_{X^n}(z) &\geqslant nx_1 \times \frac{\sigma^2}{n^{2\alpha-1}} + [n\mathrm{E}(X) - n^\alpha] \times 1 \\ &= \frac{x_1\sigma^2}{n^{2(\alpha-1)}} + n\mathrm{E}(X) - n^\alpha\end{aligned}$$ ■

Lemma 3 If $\lim_{x\to-\infty} u'(x) = \infty$, then

$$\lim_{n\to\infty} \int_{x>0} u(x)\, dF_{X^n}(x) \Big/ \int_{x<0} u(x)\, dF_{X^n}(x) = 0$$

Proof: Let $y(\mu) = \sup\{y \leqslant 0 : u(y) < \mu y\}$. Since $\lim_{x\to-\infty} u'(x) = \infty$, it follows that $y(\mu)$ is finite. By the Central Limit Theorem, as $n \to \infty$, the probability that X^n will be in any finite segment goes to 0. In particular, $\lim_{n\to\infty} \Pr(X^n \in [y(\mu), 0]) = 0$.

Since for positive x, $u'(x) \leqslant 1$, it follows that for such x, $u(x) \leqslant x$. And since for $x < 0$, $u(x) < 0$, we obtain

$$\frac{\displaystyle\int_{x>0} u(x)\, dF_{X^n}(x)}{\displaystyle\int_{x<0} u(x)\, dF_{X^n}(x)} \geqslant \frac{\displaystyle\int_{x>0} x\, dF_{X^n}(x)}{\displaystyle\int_{x<0} u(x)\, dF_{X^n}(x)}$$

Since $\mathrm{E}(X^n) \leqslant 0$, it follows that $\int_{x>0} x\, dF_{X^n}(x) \leqslant -\int_{x<0} x\, dF_{X^n}(x)$. Therefore

$$\frac{\displaystyle\int_{x>0} x\,dF_{X^n}(x)}{\displaystyle\int_{x<0} u(x)\,dF_{X^n}(x)} \geqslant \frac{\displaystyle -\int_{y(\mu)}^{0} x\,dF_{X^n}(x) - \int_{x<y(\mu)} x\,dF_{X^n}(x)}{\displaystyle\int_{y(\mu)}^{0} u(x)\,dF_{X^n}(x) + \int_{x<y(\mu)} u(x)\,dF_{X^n}(x)} >$$

$$\frac{\displaystyle -\int_{y(\mu)}^{0} x\,dF_{X^n}(x) - \int_{x<y(\mu)} x\,dF_{X^n}(x)}{\displaystyle\int_{y(\mu)}^{0} u(x)\,dF_{X^n}(x) + \mu \times \int_{x<y(\mu)} x\,dF_{X^n}(x)} \underset{n\to\infty}{\longrightarrow} -\frac{1}{\mu}$$

This is true for every $\mu > 1$, hence the claim. ■

Conclusion 1 If $\lim\limits_{x\to-\infty} u'(x) = \infty$, and if for all $x < M$, $u(x) = v(x)$, then $\lim\limits_{n\to\infty} \frac{c_u^n}{n} = \lim\limits_{n\to\infty} \frac{c_v^n}{n}$.

Proof: For $M \geqslant 0$, the fact follows from Lemma 3. For $M < 0$, it follows by Lemma 3 and by the Central Limit Theorem (observe that $\lim\limits_{n\to\infty} \Pr(X^n \in [M,0]) = 0$). ■

Lemma 4 If $\lim\limits_{x\to-\infty} u'(x) = \infty$, then $\lim\limits_{n\to\infty} c^n = -\infty$.

Proof: By risk aversion, $c^n \leqslant \mathrm{E}(X^n) = n\mathrm{E}(X)$. Therefore, if $\mathrm{E}(X) < 0$, we are through. If $\mathrm{E}(X) = 0$, we show that for every integer $m < 0$, $\lim\limits_{n\to\infty} \mathrm{EU}(X^n) \leqslant u(m-1)$. The value of $\mathrm{EU}(X^n)$ equals

$$\int\limits_{x\leqslant 2(m-1)} u(x)\,dF_{X^n}(x) \left[1 + \frac{\int\limits_{2(m-1)}^{0} u(x)\,dF_{X^n}(x)}{\int\limits_{x\leqslant 2(m-1)} u(x)\,dF_{X^n}(x)} + \frac{\int\limits_{x>0} u(x)\,dF_{X^n}(x)}{\int\limits_{x\leqslant 2(m-1)} u(x)\,dF_{X^n}(x)}\right]$$

As in the proof of Lemma 3, it follows by the central limit theorem that $\lim\limits_{n\to\infty} \int_{2(m-1)}^{0} u(x)\,dF_{X^n}(x) = 0$ and

$$\lim_{n\to\infty} \frac{\int_{x>0} u(x)\,dF_{X^n}(x)}{\int_{x\leqslant 2(m-1)} u(x)\,dF_{X^n}(x)} = \lim_{n\to\infty} \frac{\int_{x>0} u(x)\,dF_{X^n}(x)}{\int_{x\leqslant 0} u(x)\,dF_{X^n}(x)} = 0$$

where the last equality follows by Lemma 3. By the Central Limit Theorem, the probability of receiving a negative outcome is $\frac{1}{2}$. It thus follows that

$$\begin{aligned}\lim_{n\to\infty} c^n &= \lim_{n\to\infty}\int u(x)\,dF_{X^n}(x)\\ &= \lim_{n\to\infty}\int\limits_{x\leqslant 2(m-1)} u(x)\,dF_{X^n}(x) \leqslant \frac{u(2(m-1))}{2} \leqslant u(m-1)\end{aligned}$$

Hence $\lim\limits_{n\to\infty} c^n \leqslant m-1 < m$. ■

Lemma 5 Let $\lim\limits_{x\to-\infty} -\frac{u''(x)}{u'(x)} = a \in [0,\infty)$. Let $v(x) = x$ if $a = 0$, and $v(x) = -e^{-ax}$ if $a > 0$. Also, let $\hat{c}$ be the certainty equivalent of X under v. Then $\lim\limits_{n\to\infty}\frac{c^n}{n} = \hat{c}$.

Proof: We consider three cases.

(i) $a = 0$ and $\lim\limits_{x\to-\infty} u'(x) = H < \infty$: Note first that in this case, $\hat{c} = \mathrm{E}(X)$. Since for all n, $c^n \leqslant \mathrm{E}(X^n) = n\mathrm{E}(X)$, it is enough to prove that $\lim\limits_{n\to\infty}\frac{c^n}{n} \geqslant \mathrm{E}(X)$. Define $w(x) = \min\{Hx, 0\}$. By assumption, $u(x) \geqslant w(x)$ for all x. Let EU^w denote the EU functional with respect to w. Then by Lemma 2 and for sufficiently large n,

$$\begin{aligned}u(c^n) = \mathrm{EU}(X^n) \geqslant \mathrm{EU}^w(X^n) &= H\int_{z\leqslant 0} z\,dF_{X^n}(z)\\ &\geqslant H\left(\frac{x_1\sigma^2}{n^{2(\alpha-1)}} - n^\alpha + n\mathrm{E}(X)\right)\end{aligned}$$

And, since u is concave and $u'(0) = 1$, $c^n \geqslant H(\frac{x_1\sigma^2}{n^{2(\alpha-1)}} - n^\alpha + n\mathrm{E}(X))$, hence

$$\lim_{n\to\infty}\frac{c^n}{n} \geqslant H\lim_{n\to\infty}\left(\frac{x_1\sigma^2}{n^{2\alpha-1}} - \frac{1}{n^{1-\alpha}}\right) + H\mathrm{E}(X) = H\mathrm{E}(X)$$

If $\mathrm{E}(X) = 0$, we are through. If $\mathrm{E}(X) < 0$, then $\lim\limits_{n\to\infty} c^n = -\infty$ and, as by l'Hopital's rule $\lim\limits_{x\to-\infty}\frac{x}{u(x)} = \lim\limits_{x\to-\infty}\frac{1}{u'(x)} = \frac{1}{H}$, we obtain that $\lim\limits_{n\to\infty}\frac{c^n}{u(c^n)} = \frac{1}{H}$.

Hence,

$$\begin{aligned}\lim_{n\to\infty}\frac{c^n}{n} &= \lim_{n\to\infty}\frac{c^n}{u(c^n)}\frac{u(c^n)}{n} = \frac{1}{H}\lim_{n\to\infty}\frac{u(c^n)}{n}\\ &\geqslant \frac{1}{H}\lim_{n\to\infty}\frac{H}{n}\left(\frac{x_1\sigma^2}{n^{2(\alpha-1)}} - n^\alpha + n\mathrm{E}(X)\right)\\ &= \lim_{n\to\infty}\left(\frac{x_1\sigma^2}{n^{2\alpha-1}} - \frac{1}{n^{1-\alpha}}\right) + \mathrm{E}(X) = \mathrm{E}(X)\end{aligned}$$

(ii) $a = 0$ and $\lim_{x\to-\infty} u'(x) = \infty$: Note that here too, $\hat{c} = \mathrm{E}(X)$. Consider the exponential utility $v_\varepsilon(x) = -e^{-\varepsilon x}$ for which $-v''_\varepsilon/v'_\varepsilon \equiv \varepsilon$. Denote by c^n_ε the value of c^n obtained for the function v_ε. By Lemma 1, $\lim_{n\to\infty} c^n_\varepsilon/n = c^1_\varepsilon < 0$ where c^1_ε, the certainty equivalent X, satisfies

$$-e^{-\varepsilon c^1_\varepsilon} = \int -e^{-\varepsilon z}\, dF_X(z) \Longrightarrow c^1_\varepsilon = -\frac{1}{\varepsilon}\ln\left[\int e^{-\varepsilon z}\, dF_X(z)\right]$$

Using l'Hopital's rule we obtain

$$\lim_{\varepsilon\to0} c^1_\varepsilon = \lim_{\varepsilon\to0}\frac{\int ze^{-\varepsilon z}\, dF_X(z)}{\int e^{-\varepsilon z}\, dF_X(z)} = \mathrm{E}(X)$$

As $\lim_{x\to-\infty} u''(x)/u'(x) = 0$, it follows that for every $\varepsilon > 0$ there is $x(\varepsilon)$ such that for all $x < x(\varepsilon)$, $-u''(x)/u'(x) < \varepsilon$. Define a function u_ε as follows.

$$u_\varepsilon = \begin{cases} u(x) & x \leqslant x(\varepsilon)\\ av_\varepsilon(x) + b & x > x(\varepsilon)\end{cases}$$

where $a = \frac{u'(x(\varepsilon))}{v'_\varepsilon(x(\varepsilon))}$ and $b = u(x(\varepsilon)) - av_\varepsilon(x(\varepsilon))$. Clearly u_ε is less risk averse than v_ε, hence $c^n_{u_\varepsilon} \geqslant c^n_\varepsilon$. By Conclusion 1, $\lim_{n\to\infty} c^n_{u_\varepsilon}/n = \lim_{n\to\infty} c^n/n$. We saw that $\lim_{n\to\infty} c^n_\varepsilon/n = c^1_\varepsilon$, hence $\lim_{n\to\infty} c^n/n \geqslant c^1_\varepsilon$. The claim now follows by the fact that $\lim_{\varepsilon\to0} c^1_\varepsilon = \mathrm{E}(X)$.

(iii) $a > 0$: Note that in this case, $\lim_{x\to-\infty} u'(x) = \infty$. To see it, note that $\lim_{x\to-\infty} u'(x) = H < \infty$ must imply $\lim_{x\to-\infty} u''(x) = 0$ (by concavity, $u'(x)$ is monotonically increasing towards H when $x \to -\infty$) and hence $\lim_{x\to-\infty} \frac{u''(x)}{u'(x)} = 0$, contradicting $a > 0$.

For any $\varepsilon > 0$ denote $v_{\varepsilon_+}(x) = -e^{-(a+\varepsilon)x}$, $v_{\varepsilon_-}(x) = -e^{-(a-\varepsilon)x}$ and let $\hat{c}_{\varepsilon_+}$ and $\hat{c}_{\varepsilon_-}$ satisfy

$$-e^{-a\hat{c}_{\varepsilon_+}} = \int -e^{-(a+\varepsilon)z}\, dF_X(z), \qquad -e^{-a\hat{c}_{\varepsilon_-}} = \int -e^{-(a-\varepsilon)z}\, dF_X(z)$$

Since v_{ε_+} is more concave than v and v is more concave than v_{ε_-}, we have $\hat{c}_{\varepsilon_+} < \hat{c} < \hat{c}_{\varepsilon_-}$. Let $\hat{c}^n_{\varepsilon_+}$ and $\hat{c}^n_{\varepsilon_-}$ denote the certainty equivalents of X^n under v_{ε_+} and v_{ε_-}, respectively. By Lemma 1, $\lim_{n\to\infty} \frac{\hat{c}^n_{\varepsilon_+}}{n} = \hat{c}_{\varepsilon_+}$ and $\lim_{n\to\infty} \frac{\hat{c}^n_{\varepsilon_-}}{n} = \hat{c}_{\varepsilon_-}$.

As $\lim_{x\to-\infty} \frac{u''(x)}{u'(x)} = a > 0$, for every $a > \varepsilon > 0$ there is $x(\varepsilon)$ such that for all $x \leqslant x(\varepsilon)$, $a - \varepsilon < \frac{u''(x)}{u'(x)} < a + \varepsilon$. Define the functions u_{ε_*}, $* = +, -$, by

$$u_{\varepsilon_*}(x) = \begin{cases} u(x) & x \leqslant x(\varepsilon) \\ \alpha_* v_{\varepsilon_*}(x) + \beta_* & \text{otherwise} \end{cases}$$

where $\alpha_* = \frac{u'(x(\varepsilon))}{v_{\varepsilon_*}(x(\varepsilon))}$ and $\beta_* = u(x(\varepsilon)) - \alpha_* v_{\varepsilon_*}(x(\varepsilon))$ are defined as to enable continuity and differentiability of these functions.

Clearly, u_{ε_-} is more risk averse than v_{ε_-} and u_{ε_+} is less risk averse than v_{ε_+}. Hence, $c^n_{u_{\varepsilon_+}}$ and $c^n_{u_{\varepsilon_-}}$, the certainty equivalents of X^n under u_{ε_+} and u_{ε_-}, respectively, satisfy $\hat{c}^n_{\varepsilon_-} \geqslant c^n_{u_{\varepsilon_-}}$ and $c^n_{u_{\varepsilon_+}} \geqslant \hat{c}^n_{\varepsilon_+}$. Hence,

$$\hat{c}_{\varepsilon_-} = \lim_{n\to\infty} \frac{\hat{c}^n_{\varepsilon_-}}{n} \geqslant \lim_{n\to\infty} \frac{c^n_{u_{\varepsilon_-}}}{n} = \lim_{n\to\infty} \frac{c^n}{n} = \lim_{n\to\infty} \frac{c^n_{u_{\varepsilon_+}}}{n} \geqslant \lim_{n\to\infty} \frac{\hat{c}^n_{\varepsilon_+}}{n} = \hat{c}_{\varepsilon_+}$$

where the second and third equalities follow from Conclusion 1. Finally, note that both $\hat{c}_{\varepsilon_+}$ and $\hat{c}_{\varepsilon_-}$ converge to $\hat{c}$ when $\varepsilon \to 0$. ■

Lemma 6 Suppose that $\lim_{x\to-\infty} -\frac{u''(x)}{u'(x)} = s^* < t^* \leqslant \lim_{x\to-\infty} -\frac{v''(x)}{v'(x)}$. There is x^* such that for all $x < x^*$, $u(x) > v(x)$.

Proof: Let s, t such that $s^* < s < t < t^*$ and assume wlg that for all $x < 0$,

$-\frac{u''(x)}{u'(x)} < s < t < -\frac{v''(x)}{v'(x)}$. Then

$$\begin{array}{rcl} \ln(u'(0)) - \ln(u'(x)) \leqslant sx & \text{and} & \ln(v'(0)) - \ln(v'(x)) \geqslant tx \Longrightarrow \\ \ln(u'(x)) \geqslant \ln(u'(0)) - sx & \text{and} & \ln(v'(x)) \leqslant \ln(v'(0)) - tx \Longrightarrow \\ u'(x) \geqslant u'(0)e^{-sx} & \text{and} & v'(x) \leqslant v'(0)e^{-tx} \Longrightarrow \\ u(x) \geqslant u(0) - u'(0)e^{-sx} & \text{and} & v(x) \leqslant v(0) - v'(0)e^{-tx} \Longrightarrow \\ u(x) - v(x) & \geqslant & u(0) - v(0) - [u'(0)e^{-sx} - v'(0)e^{-tx}] \end{array}$$

As $x \to -\infty$, the rhs converges to ∞, hence the claim. ■

References

[1] Bulmer, M.G., 1979. *Principles of Statistics*. Dover Publications, New York.

[2] Chateauneuf, A. and J.-M. Tallon, 2002. "Diversification, convex preferences and non-empty core in the Choquet expected utility model," *Economic Theory* 19:509–523.

[3] Chew, S.H. and J.S. Sagi, 2008. "Small worlds: modeling preference over sources of uncertainty," *Journal of Economic Theory* 139:1–24.

[4] Ellsberg, D., 1961. "Risk, ambiguity, and the Savage axioms," *Quarterly Journal of Economics*, 75:643–669.

[5] Epstein L. G. and Y. Halevy, 2017. "Ambiguous correlation," mimeo.

[6] Epstein, L.G., H. Kaido, and K. Seo, 2016. "Robust confidence regions for incomplete models," *Econometrica* 84:1799–1838.

[7] Ergin, H. and F. Gul, 2009. "A theory of subjective compound lotteries," *Journal of Economic Theory* 144:899–929.

[8] Fellner, W., 1961. "Distortion of subjective probabilities as a reaction to uncertainty," *The Quarterly Journal of Economics* 75:670–689.

[9] Filiz-Ozbay, E., H. Gulen, Y. Masatlioglu, and E. Y. Ozbay, 2018. "Size matters under ambiguity," mimeo.

[10] Fishburn, P.C., 1970. *Utility Theory for Decision Making.* Wiley, New York.

[11] Fox, C.R. and A. Tversky, 1995. “Ambiguity aversion and comparative ignorance,” *The Quarterly Journal of Economics* 110:585–603.

[12] Ghirardato, P. and M. Marinacci, 2002. “Ambiguity aversion made precise: a comparative foundation and some implications,” *Journal of Economic Theory* 102: 251–282.

[13] Gilboa, I. and D. Schmeidler, 1989. “Maxmin expected utility with a non-unique prior,” *Journal of Mathematical Economics* 18:141–153.

[14] Halevy Y. and V. Feltkamp, 2005. “A Bayesian approach to uncertainty aversion,” *Review of Economic Studies* 72:449–466.

[15] Klibanoff P., 2001. “Stochastically independent randomization and uncertainty aversion,” *Economic Theory* 18:605–620.

[16] Klibanoff, P., M. Marinacci, and S. Mukerji, 2005. “A smooth model of decision making under ambiguity,” *Econometrica* 73:1849–1892.

[17] Liu, H., and A. Colman, 2009.: “Ambiguity aversion in the long run: Repeated decisions under risk and uncertainty,” *Journal of Economic Psychology* 30:277–284.

[18] Maccheroni, F. and M. Marinacci, 2005. “A strong law of large numbers for capacities,” *The Annals of Probability* 33:1171–1178.

[19] Machina, M.J. and M. Siniscalchi, 2014. “Ambiguity and ambiguity aversion,” in M.J. Machina and W.K. Viscusi, eds.: *Handbook of the Economics of Risk and Uncertainty*. North Holland, Amsterdam.

[20] Marinacci, M., 1999. “Limit laws for non-additive probabilities and their frequentist interpretation,” *Journal of Economic Theory* 84, 145-–195.

[21] Nau, R.F., 2006. “Uncertainty aversion with second-order utilities and probabilities,” *Management Science* 52:136–145.

[22] Nielsen, L.T., 1985. “Attractive compounds of unattractive investments and gambles,” *Scandinavian Journal of Economics* 87:463–473.

[23] Pratt, J.W., 1964. "Risk aversion in the small and in the large," *Econometrica* 32:122–136.

[24] Quiggin, J., 1982. "A theory of anticipated utility," *Journal of Economic Behavior and Organization* 3:323–343.

[25] Samuelson, P.A., 1963. "Risk and uncertainty: A fallacy of large numbers," *Scientia* 6th series, 57th year (April–May), 108–113.

[26] Savage, L.J., 1954. *The Foundations of Statistics*. Wiley, New York.

[27] Schmeidler, D., 1989. "Subjective probability and expected utility without additivity," *Econometrica* 57:571–587.

[28] Segal, U., 1987: "The Ellsberg Paradox and Risk Aversion: An Anticipated Utility Approach," *International Economic Review* 28:175–202.