

Sequential Formation of Alliances in Survival Contests

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Abstract

We consider a *sequential formation* of alliances à la Bloch (1996) and Okada (1996) followed by a two-stage contest in which alliances first compete with each other, and then the members in the winning alliance compete again for an indivisible prize. In contrast to Konishi and Pan (2019) which adopted an *open-membership* game as the alliance formation process, alliances are allowed to limit their memberships (excludable alliances). We show that if members' efforts are strongly complementary to each other, there will be exactly two asymmetric alliances—the larger alliance is formed first and then the rest of the players form the smaller one. This result contrasts with the one under open membership, where moderate complementarity is necessary to support a two-alliance structure. It is also in stark contrast with Bloch et al. (2006), where they show that a grand coalition is formed in the same game if the prize is divisible and a binding contract is possible to avoid further conflicts after an alliance wins the prize.

1 Introduction

In their influential paper, Esteban and Sákovics (2003) consider a three-person strategic alliance formation in a Tullock contest model in which players compete for an indivisible prize, and demonstrate that an alliance involves strategic disadvantages (see also Konrad 2009). There are two main disadvantageous forces against forming an alliance: First, if an alliance is formed, there will be

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an additional contest that dissipates the members' rents even if the alliance wins the first race. Because of this *rent-dissipation* effect, the members of the alliance have lower valuations for winning in the first race, reducing their efforts and the winning probability. Second, even without the rent-dissipation problem, if the winning prize is shared equally, there are still *free-riding* incentives for the alliance members to reduce efforts, and consequently, the winning probability. As a result, they conclude that it is hard to materialize strategic alliances in a Tullock contest model.^{1,2} Konrad (2009) points out that these disincentive effects are not specific to Tullock contest models—they also appear in first price all-pay auctions.

In a companion paper, Konishi and Pan (2019), we provide a simple solution for this alliance paradox by using a CES effort aggregator function to introduce complementarity in efforts (see Kolmer and Rommeswinkel 2013).³ We assume that each individual member's marginal effort cost is constant in order to limit the benefits of forming an alliance to effort complementarity only. In that paper, we model an alliance formation process as an *open-membership* coalition formation game. In stage 1, players form alliances by an open-membership game (see Yi, 1997, and Bogomolnaia and Jackson, 2002). In stage 2, alliances compete in a contest with each other, and in stage 3, the winning alliance members compete in the standard Tullock contest for the indivisible prize. We show that when the complementarity parameter in CES function is small, there are spin-off incentives for alliance members, while when the complementarity parameter is large, players want to join a bigger alliance,

¹Konrad (2004) considers an asymmetric all-pay auction game with exogenously determined hierarchical tournament structure, and shows that the highest valuation player might not have a chance to become the final winner depending on the hierarchical structure. In contrast, Konrad (2012) consider an alliance formation problem in the case where players with homogeneous valuations play an all-pay auction game while their budgets for bidding are private information. He shows that alliances always have merging incentives, and the grand alliance emerges.

²Wärneryd (1998) shows that forming alliances and competing in a multi-stage competition reduce wasteful competition and increase total welfare. This resource saving effect is difficult to realize due to the disadvantageous effect on alliances when members' individual efforts are perfectly substitutable.

³Complementarity in efforts within a group in Esteban and Ray (2011) is more subtle. They analyze the conflict between two ethnic groups by assuming that players have heterogeneous opportunity costs of financial and human opportunity costs, and they can contribute financially to a conflict or they can directly participate as activists. They show that opportunity cost heterogeneity in a group increases the level of conflicts. Their result can be interpreted that an increase in complementarity within groups intensifies group competition.

and end up with a trivial grand alliance.⁴ They show that for intermediate values of the CES complementarity parameter, there exists a unique nontrivial two-alliance equilibrium.

In contrast, in this paper, we use Bloch’s (1996) and Okada’s (1996) *sequential* coalition (alliance) formation game (along the line of a noncooperative coalition bargaining game in Chatterjee, et al. 1993). Although the open-membership game in Konishi and Pan (2019) is widely used in coalition formation games, the *non-excludability* – that is, players are allowed to freely choose their alliance without being excluded – may not reflect the nature of alliance formation in situations such as a local public good economy.

The results in open-membership and sequential coalition formation games are quite different. In an open-membership game, if effort complementarity is higher than a critical value, belonging to a larger alliance becomes strongly preferable, despite the fact that there will be negative congestion effects, which encourages all players to form a grand coalition. This is a prisoners’ dilemma phenomenon. In contrast, with sequential coalition formation, a coalition is able to avoid becoming too large, although it also needs to think about the response from the rest of the players in their strategic interactions. Somewhat interestingly, there will again be two alliances in equilibrium, but for this we need strong effort complementarity. Note that due to excludability, even if complementarity is very strong, the grand alliance will not emerge in equilibrium. Thus, although the two-alliance result seems similar to the one in the open-membership case, they have no obvious relationship. Indeed, the parameter ranges to have two-equilibrium results in these two games have no intersection with each other, and two coalitions are similar in their sizes in the open-membership game, but are quite asymmetric in the sequential coalition formation game. We further show that the first alliance is larger than the second, and the members of the former receive higher payoffs than the latter. This property assures that the alliance structure is robust in the protocol: that is, we obtain the same alliance structure in Bloch’s deterministic protocol and in Okada’s random protocol. Note that Bloch (1997) and Yi (1997) provide a set of sufficient conditions under which two coalitions are formed in a sequential alliance formation game, but these conditions and our conditions are independent of each other. Moreover, we get the two-alliance result only when effort complementarity is large enough.

We also provide numerical examples for different values of the CES effort

⁴Given the way we set up the multi-stage game, a singleton-only alliance structure and a grand alliance structure are practically identical, since the former does not have the third stage competition, and the latter does not have the second stage competition.

complementarity parameter under a small number of players (ten players). We show that there will be no alliance if σ is small, but as σ goes up the sizes of alliances increase. Once σ passes a certain threshold value, there will be only two (asymmetric) alliances in equilibrium, and every player participates in alliances as we have shown in our main theorem.

The rest of the paper is organized as follows. In the next subsection, we review the relevant literature. Section 2 introduces the model, and Sections 3 and 4 investigate subgames in stages 3 and 2, respectively. Section 5 presents results on equilibrium alliance structures, and Section 6 provides numerical examples. Section 7 concludes.

1.1 Literature Review

Since we provide a general literature review in our companion paper (Konishi and Pan 2019), we will concentrate on the games that determine an alliance structure. In the companion paper, we used so-called open-membership game where all players can move freely without being excluded from alliances.⁵ However, depending on the nature of alliances we consider, we may want to see how equilibrium alliance structure is affected by allowing exclusive memberships of alliances.

Although we can think of different ways to introduce "excludability" of alliance memberships in an alliance formation game (see Hart and Kurz 1983, and Bloch 1997), the most popular way in the literature is to extend Rubinstein's two-person noncooperative bargaining game to a sequential coalition formation game: Chatterjee et al. (1993), Bloch (1996), Okada (1996), and Ray and Vohra (1999), among others. Although their games differ in the methods of choosing the proposers (following different protocols), the procedures for forming coalitions are the same. At each stage, a proposer proposes a coalition she belongs to, and ask the members of the coalition whether or not they accept the offer. If every member accepts the offer, then the coalition is formed, and the leftover players continue to form coalitions by the same procedure. If any of the members of a proposed coalition rejects the offer, the coalition is not formed, and a new proposer is specified by the protocol.

In the context of contests, Bloch et al. (2006) generalize the model substantially to analyze the stability of the grand alliance in different alliance formation games, including a sequential coalition formation game in Bloch (1996). Sánchez-Páges (2007a) explores different types of stability concepts

⁵Baik and Lee (1997, 2001) use open-membership games to describe alliance formation in endogenizing the alliance structure in Nitzan's (1991) game with endogenous group sharing rules.

including sequential coalition formation games in alliance formation in contests where efforts are perfect substitutes. Sánchez-Páges (2007b) considers various stability concepts in a model where players allocate endowment into productive and exploitive activities. These papers assume the award is divisible, and alliance members can write a binding contract for sharing rules in the case of the alliance’s winning.⁶ In contrast, in our paper, we do not allow for any side payment, and players cannot credibly commit to any intra-alliance distribution rule as in Esteban and Sákovics (2003). We will focus on the case where the reward is indivisible so that conflict cannot be eliminated until the ultimate winner is determined.

2 The Model

There are N players who get an indivisible prize (say, to be the head of an organization). There is no side payment allowed. The set of players is also denoted by $N = \{1, \dots, N\}$, and they can form alliances exclusively for the purpose of being elected. Each player $i \in N$ can make an effort to enhance the popularity of her alliance and that of herself. We assume that each player has an identical linear cost function $C(e_i) = e_i$ for all $e_i \geq 0$.

We consider a three-period dynamic contest game *preceded by an alliances formation process*: in the first period, players’ form alliances endogenously. In the second period, the alliances compete to decide the winning alliance, and then the players in the winning alliance compete with each other to determine the final winner in the third period.

We model the sequential alliance formation process as the following: in period 1, there are rounds $j = 1, 2, \dots$. At every round $j = 1, 2, \dots$, one player is selected as a proposer with equal probability among all players still active in the round j . Let N_j be the set of all active players at round j , where $N_1 = N$. The selected player i proposes a coalition S_j with $i \in S_j \subseteq N_j$. All other players in S_j either accept or reject the proposal sequentially. We assume that the responses are made according to a predetermined order over N_j (the order of responses does not affect our results). If all other players in the coalition S_j accept the proposal, then it is agreed upon and enforced, and the remaining players in $N_{j+1} = N_j \setminus S_j$ can continue negotiations in the next round $j + 1$. If some members in S_j reject the offer, round r ends and negotiations go on to the next round and a new proposer is randomly selected from $N_{j+1} = N_j$ by the same rule. Following a coalitional bargaining model in Ray and Vohra (1999), we assume that whenever an offer is rejected, some amount of time passes

⁶Our game has effort complementarity in the model, which is an additional difference.

and a time discount factor $\delta \in (0, 1)$ applies to the final payoff. The process continues until there is no player left and $\pi = \{S_1, S_2, \dots, S_J\}$ is formed.

We introduce potential benefits for players who belong to an alliance—complementarity in aggregating efforts by all alliance members. That is, if player i belongs to alliance $j = 1, \dots, J$ with $S_j \subset N$ as the set of members, and these members make efforts $(e_{hj})_{h \in S_j}$, then the aggregated effort of alliance j , E_j , is described by a CES aggregator function

$$E_j = \left(\sum_{h \in S_j} e_{hj}^{1-\sigma} \right)^{\frac{1}{1-\sigma}}, \quad (1)$$

where $\sigma \in (0, 1]$ is a parameter that describes the degree of complementarity: if $\sigma = 0$ it is a linear function, and if $\sigma = 1$ it is a Cobb-Douglas function. Thus, as σ goes up, the complementarity of members' efforts increases.

Candidate i in alliance j decides how much effort e_{ij} to contribute to her alliance j . The winning probabilities of an alliance is a Tullock-style contest. That is, an alliance j 's “winning probability” given its members' efforts is

$$p_j = \frac{E_j}{\sum_{k \in J} E_k}. \quad (2)$$

An indivisible prize is valued as $V > 0$, which is common to all players. Since the prize is indivisible, one player in the winning alliance in the second stage must be selected as the final winner in the third-stage contest.

In the third-stage competition, we assume that a Tullock contest takes place within the winning alliance S_j . Denoting the second-stage effort as \hat{e}_i , the winning probability of player $i \in S_j$ is

$$p_i = \frac{\hat{e}_i}{\sum_{h \in S_j} \hat{e}_h} \quad (3)$$

Formally, an alliance structure is a partition of the set of players N , $\pi = \{S_1, \dots, S_J\}$, where each alliance j consists of a set of players S_j and $\cup_{j \in J} S_j = N$, and $S_{j'} \cap S_j = \emptyset$ for any $j, j' \in \{1, \dots, J\}$ with $j \neq j'$. Since we assume that players are ex-ante homogenous, we also call $\{n_1, \dots, n_J\}$ an alliance structure with $n_j = |S_j|$ for all $j = 1, \dots, J$. Our three-stage dynamic contest game with sequential alliance formation is summarized as:

Stage 1. In round $j = 1, 2, \dots$, one player is selected as a proposer with equal probability among all active players in the round j , N_j , where $N_1 = N$.⁷

⁷This is the random proposer protocol put forth by Okada (1996). Bloch (1996) uses a deterministic protocol, but the results we obtain in these two setups are the same if effort complementarity is high enough.

The selected player proposes an alliance $S_j \subseteq N_j$. All other players in S_j either accept or reject the proposal sequentially. If all other players in the alliance S_j accept the proposal, S_j is formed and removed from the process, and $j+1$ round starts with the remaining players $N_{j+1} = N_j \setminus S_j$. Otherwise, payoff discounts by $\delta \in (0, 1)$ apply to all players, the round $r+1$ starts with $N_{j+1} = N_j$ by the same rule. The process continues until there is no player left and $\pi = \{S_1, S_2, \dots, S_J\}$ is formed.⁸

- Stage 2. All players $i \in N$ choose effort $e_i \in \mathbb{R}_+$ simultaneously, knowing the aggregated effort of her alliance is (1). The inter-alliance contest is a Tullock contest with winning probabilities equal to (2).
- Stage 3. All members of the winning alliance S_j choose effort $\hat{e}_i \in \mathbb{R}_+$ simultaneously. The ultimate winner is selected by a simple Tullock contest with winning probabilities equal to (3).

We use standard subgame perfect Nash equilibrium as the solution of this dynamic game. We consider equilibria in pure strategies only. We will analyze this game by backward induction.

3 Equilibrium

3.1 Stage 3: Final Contest within the Winning Alliance

In the third stage, all members in the winning alliance S_j in the first stage engage in a Tullock contest by exerting effort $\hat{e}_i \geq 0$. Thus, player i 's winning probability is

$$p_i = \frac{\hat{e}_i}{\sum_{h \in N_j} \hat{e}_h}.$$

For any player i in the winning group j , the expect payoff in stage 3 is

$$\tilde{V}_i = \frac{\hat{e}_i}{\hat{e}_i + \sum_{h \neq i} \hat{e}_h} V - \hat{e}_i$$

The first-order condition implies that

$$\frac{1 - p_i}{\hat{e}_i + \sum_{h \neq i} \hat{e}_h} - 1 = 0 \Rightarrow \frac{1}{\hat{e}_i} p_i (1 - p_i) V - 1 = 0$$

⁸Given the alliances formed in the previous rounds, the remaining players may be forced to be inactive even when $S_j = N_j$ at round j . In such cases, we assume that they form the coalition $S_j = N_j$ to close the alliance formation process without loss of generality.

Since players are homogeneous, $p_i(1 - p_i) = \frac{n_j - 1}{n_j^2}$ is the same for all i in the winning group j . Then, we have the following proposition.

Proposition 1. *Suppose that the winning alliance of the first stage has size n_j . Then, the second-stage equilibrium strategy and payoff are*

$$\hat{e}^j = \frac{n_j - 1}{n_j^2}V \text{ and } \tilde{V}^j = \frac{V}{n_j} \left(1 - \frac{n_j - 1}{n_j}\right) = \frac{V}{n_j^2}$$

3.2 Stage 2: Contest between Alliances

Consider an inter-alliance contest problem. Without loss of generality, we reorder any alliance structure π from the first stage so that $n_1 \geq n_2 \geq \dots \geq n_{J^*}$. From Proposition 1, we know that for a given size of alliance n_j the payoff of intra-alliance contest is determined by $\tilde{V}_j = \frac{V}{n_j^2}$. In the companion paper, Konishi and Pan (2019) have the following result.

Theorem 1. (Konishi and Pan, 2019) *There exists a unique equilibrium in the second stage for any partition of players $\pi = \{n_1, \dots, n_{J^*}\}$ characterized by $j^* \in \{1, \dots, J^*\}$ such that $p_j^* > 0$ (active alliance) for all $j \leq j^*$, while $p_j^* = 0$ (inactive alliance) for all $j > j^*$. Moreover, the members of alliance $j = 1, \dots, J^*$ obtain payoff*

$$u_j = \begin{cases} \frac{1}{n_j^2} \left[1 - (j^* - 1) \frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}} \right] \left[1 - (j^* - 1) \frac{n_j^{\frac{1-2\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}} \right] & \text{if } j \leq j^* \\ 0 & \text{if } j > j^* \end{cases}$$

by exerting effort

$$e_j = \begin{cases} \frac{1}{n_j^{\frac{1}{1-\sigma}}} \left[1 - (j^* - 1) \frac{n_j^{\frac{2-3\sigma}{1-\sigma}}}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}} \right] \frac{(j^* - 1)V}{\sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}} & \text{if } j \leq j^* \\ 0 & \text{if } j > j^* \end{cases}$$

and

$$(j^* - 1)n_j^{\frac{2-3\sigma}{1-\sigma}} < \sum_{j'=1}^{j^*} n_{j'}^{\frac{2-3\sigma}{1-\sigma}}$$

holds for all $j = 1, \dots, j^*$.

3.3 Stage 1: Alliance Structures under Sequential Coalition Formation

Here, we consider a sequential coalition formation game with exclusive alliances a la Bloch (1996) and Okada (1996). The main results are as follows.

Theorem 2. *For any N , there is $\bar{\sigma}(N)$ such that, for all $\sigma \geq \bar{\sigma}(N)$, there are only two alliances in equilibrium. All players belong to one of the two alliances, and the first alliance to move is larger in size than the second one.*

We prove this theorem by a sequence of lemmas. For analytical convenience, we will consider fractions of alliance sizes. Let $x_j = \frac{n_j}{N}$, where N is the total population. Players' optimization problems in forming alliances can be described as a fractional alliance choice problem. For notational simplicity, let

$$\tau = -\frac{2 - 3\sigma}{1 - \sigma} = \frac{\sigma}{1 - \sigma} - 2$$

That is, $\tau \geq 0$ if and only if $\sigma \geq \frac{2}{3}$. As σ increases to its upperbound $\sigma = 1$, τ goes to infinity. This means that if a statement holds for all $\tau > \bar{\tau}$, then there is $\bar{\sigma}$ such that the same statement holds for all $1 > \sigma > \bar{\sigma}$.

Since we consider a subgame perfect equilibrium in the sequential alliance formation game, we will start with the last alliance's size decision when there are $J \geq 1$ existing alliances with sizes (x_1, \dots, x_J) and a relatively small fraction of players left uncommitted. Let

$$\begin{aligned} \frac{1}{\bar{x}^\tau} &\equiv \frac{1}{J} \sum_{j=1}^J \frac{1}{x_j^\tau} \\ \Rightarrow \bar{x} &= \frac{1}{\sqrt[\tau]{\frac{1}{J} \sum_{j=1}^J \frac{1}{x_j^\tau}}} \end{aligned}$$

That is, \bar{x} is the "power average" of the sizes of J alliances. Let $u(x, \bar{x}; J + 1)$ be size x alliance's payoff when the average size of all other alliances is \bar{x} when there are $J + 1$ active alliances including a size x alliance. This alliance's payoff can be written as

$$u(x, \bar{x}; J + 1) = \frac{\left(J \frac{1}{\bar{x}^\tau} - (J - 1) \frac{1}{x^\tau}\right) \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau} - \frac{J}{N} \frac{1}{x^{\tau+1}}\right)}{N^2 x^2 \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right)^2}$$

Then, $u(x, \bar{x}; J + 1)$ can be written as a product of the following two functions

$$\frac{1}{N^2} f(x, \bar{x}; J + 1) = \frac{1}{N^2} \frac{\left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau} - \frac{J}{N} \frac{1}{x^{\tau+1}}\right)}{\left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right)} = \frac{1}{N^2} \left[1 - \frac{\frac{J}{N} \frac{1}{x^{\tau+1}}}{\left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right)} \right]$$

and

$$g(x, \bar{x}; J+1) = \frac{\left(J \frac{1}{\bar{x}^\tau} - (J-1) \frac{1}{x^\tau}\right)}{x^2 \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right)}$$

We have the following result.

Lemma 1. *Suppose that $J \geq 1$ alliances with their average size \bar{x} have been formed and remain active even with the entry of the $J+1$ alliance. Then, (i) $\frac{\partial u(x, \bar{x}; J+1)}{\partial \bar{x}} < 0$ for all x and \bar{x} , and (ii) $\frac{\partial u(x, \bar{x}; J+1)}{\partial x} > 0$ holds for all $\frac{J-1}{J} \bar{x} \leq \left(\frac{x}{\bar{x}}\right)^\tau \leq \frac{(2+\tau)J-4}{2J}$ when $\tau \geq 2$. Moreover, if $\left(\frac{x}{\bar{x}}\right)^\tau < \frac{J-1}{J}$, then even if the $J+1$ th alliance with size x enters, it cannot be active.*

The implications of this lemma are listed in the following corollaries.

Corollary 1. *When $\tau > \frac{4}{J}$, then the best response of the $J+1$ th alliance satisfies $x > \bar{x}$ knowing that there will be no more alliances formed after the alliance.*

Corollary 2. *Let $J = 1$ (only one alliance with size \bar{x} has been formed). Then, (i) $\frac{\partial u(x, \bar{x}; 1)}{\partial \bar{x}} < 0$ for all x and \bar{x} , and (ii) $\frac{\partial u(x, \bar{x}; 1)}{\partial x} > 0$ holds for all $\left(\frac{x}{\bar{x}}\right)^\tau \leq \frac{\tau-2}{2}$ when $\tau \geq 2$. The result (ii) implies that if $\tau > 4$, then the best response of the second alliance satisfies $x > \bar{x}$ knowing that there will be no more alliances formed after the second alliance.*

When τ is large enough, we can assure the following.

Lemma 2. *There is $\bar{\tau}(N)$ such that for all $\tau > \bar{\tau}(N)$, there are at most two different size active alliances, or all alliances are the same size.*

The following is a purely technical lemma.

Lemma 3. *Consider two allocations: $\pi = (x_1, x_2)$ and $\pi' = (x_1, \underbrace{x_2, \dots, x_2}_J)$. A size- x_2 alliance member prefers π to π' .*

The following lemma slightly strengthens the implication of Lemma 1 when τ is large enough.

Lemma 4. *Suppose that among J formed alliances, $J^M \geq 1$ of them have the largest size x^M , and $x^M < 1 - \sum_{j=1}^J x_j < 2x^M$. For all $\tau \geq \hat{\tau}(N)$, we have $u(x^M + \frac{1}{N}, x^M; J^M) > u(x, x^M; J^M)$ for all $x \leq x^M$.*

Lemma 5. *Suppose that $J = 2$. For $\tau \geq \check{\tau}(N)$ for some $\check{\tau}(N)$, $u(x, x + \frac{1}{N}; 1) < u(\frac{1}{2} + \frac{1}{N}, \frac{1}{2} - \frac{1}{N}; 1)$ for any $\frac{1}{2} > x$. That is, the benefits from belonging to a larger*

alliance with a higher winning probability dominates the loss from sharing with a larger group.

Lemma 6. *Suppose that among J formed alliances, $J^M \geq 1$ of them have the largest size x^M , and $x^M < 1 - \sum_{j=1}^J x_j < 2x^M$. For $\tau \geq \tilde{\tau}(N)$ for some $\tilde{\tau}(N)$, we have $u(x^M + \frac{1}{N}, x^M; J^M) > u(x, x^M; J^M)$ for all $x \leq x^M$, and $u(x^M, x^M + \frac{1}{N}; J^M) < u(\frac{1}{2}(1 - \sum_{j=1}^J x_j) + \frac{1}{N}, \frac{1}{2}(1 - \sum_{j=1}^J x_j) - \frac{1}{N}; 1)$. That is, the benefits of belonging to a larger alliance with a higher winning probability dominates the losses of sharing with a larger group.*

Proof of Theorem 2. We can rename $\bar{\tau}(N)$ by the maximum of the original $\tau(N)$, $\hat{\tau}(N)$, $\check{\tau}(N)$, and $\tilde{\tau}(N)$. Let $\bar{\sigma}(N)$ be σ that corresponds to $\bar{\tau}(N)$. By the sequence of the lemmas above, we consider the second mover's best or better responses.

1. Suppose that $x_1 \geq \frac{1}{2}$. By Lemma 1, $x_2 = 1 - x_1$ is the best response.
2. Suppose that $\frac{1}{3} \leq x_1 < \frac{1}{2}$. Suppose that $x_2 \leq \frac{1-x_1}{2}$. We will show that forming multiple same-size alliances is dominated by forming an alliance of size $x_1 + \frac{1}{N}$. Suppose that two or more size- x_2 alliances are formed after a size- x_1 alliance. In this case, $x_2 \leq x_1$ holds. By Lemma 3, having only one size- x_2 alliance is generally better than forming multiple of them. Since $x_2 \leq x_1$, calling x_2 is dominated by calling x_1 by Lemma 1. But Lemma 4 suggests that for the second mover calling $x_1 + \frac{1}{N}$ dominates calling x_1 , since Lemma 2 implies that there will be only two active alliances if $x_1 + \epsilon$ is called. Note, however, that even if $x_2 = x_1 + \frac{1}{N}$, the first alliance can do better by choosing size $\frac{1}{2} + \epsilon$ from the first place by Lemma 5. Thus, this case cannot be an equilibrium.
3. Suppose that $x_1 < \frac{1}{3}$. By Lemma 2, $x_2 \geq x_1$ holds (otherwise, alliance 2 will be inactive by $x_3 > x_1$). We only need to consider the case where alliance 2 calls the same size $x_2 = x_1$, which is the only possible case for alliance 1 to call a size x_1 alliance. (If $x_2 \geq x_1 + \frac{1}{N}$, alliance 3 will call size $x_3 \geq x_2$ if possible, which makes alliance 1 inactive, and otherwise, $x_3 = x_1 < x_2 + \frac{1}{N}$, and the argument in lemma 6 applies. Either way, alliance 1 does not have an incentive to call a size x_1 alliance unless $x_2 = x_1$.) A similar argument applies with $j \geq 3$, and the only possible allocation achievable is an equal division $x_j = \frac{1}{j}$ for all $j = 1, \dots, J \geq 4$. However, if so, the $J - 1$ th alliance can call $\frac{1}{j} + \frac{1}{N}$ to improve its payoff (Lemma 4), so this allocation is not achievable. Indeed, by foreseeing

this behavior by the $J - 1$ th alliance, the $J - 2$ th alliance can call a little more than one half of the set of players who do not belong to alliances 1 to $J - 3$ (Lemma 6). Then, only the $J - 2$ th and the $J - 1$ th alliances will remain active, and alliance 1 gets zero payoff (the $J - 1$ th alliance is formed by all of the rest of the players by Lemma 1). Thus, this case cannot be an equilibrium as well.

Hence, only case 1 can happen in equilibrium, and there are only two alliances in equilibrium, all players belong to one of the alliances, and the first alliance is larger than the second. \square

Remark. Since $x_1 > x_2$ holds with $u(x_1, x_2; 2) > u(x_2, x_1; 2)$ in equilibrium, there will not be any delay in forming coalitions. That is, the same outcome would realize independent of the protocol.

4 Examples with Small Population

For our analytical result, we will consider the cases of relatively low complementarity parameter with a small number of players $N = 10$. The complementarity parameter value $\sigma \geq \frac{6}{7}$ is sufficient to assure that the second mover tries to form a larger alliance than the first (Corollary 2: $\tau = \frac{\sigma}{1-\sigma} - 2$). We will consider the first mover's choices in order, and find the equilibrium outcome.

4.1 Case 1: $\sigma = \frac{6}{7}$ or $\tau = 4$

1. The first mover calls a size 5 alliance. In this case, the second mover forms another size 5 alliance by Corollary 2. Their payoffs for (5, 5) are (0.018, 0.018). Indeed, for the second mover, calling a smaller alliance is not appealing: (0.026728, 0.01494) for (5, 4), and (0.034598, 0.0089861) for (5, 3).
2. The first mover calls a size 6 alliance. The rest of players stick to each other. Their payoffs for (6, 4) are (0.022558, 0.0081571).
3. The first mover calls a size 7 alliance. The rest of the players stick to each other. Their payoffs for (7, 3) are (0.01965, 0.0024569).
4. The first mover calls a size 4 alliance. In this case, if the second mover calls a size 5 alliance, their payoffs for (4, 5) are (0.012521, 0.028641). If a size 4 alliance is called by the second mover, then the leftover two players are forced to be inactive, and the first two alliances' payoffs (4, 4) are

(0.027344, 0, 0.027344). If the second alliance calls a size 3 alliance, then the third alliance will be size 3, and their payoffs for (4, 3, 3) are (0.042323, 0.010809, 0.010809). Thus, the second mover will call a size 5, and the payoffs for (4, 5) are (0.012521, 0.028641).

5. The first mover calls a size 3 alliance. If the second mover calls a size 3, then the rest form a size 4, and this is not beneficial for the second mover (see above). If she calls a size 4, then (3, 4, 3) realizes with (0.010809, 0.042323, 0.010809). If she calls a size 5, then (3, 5) realizes, leaving an inactive size 2 alliance with payoffs (0.0089861, 0.034598). So, her best response is to call a size 4 alliance.
6. The first mover calls a size 2 alliance. Then, the second mover calls a size 5 alliance, making the first mover's alliance inactive. The payoffs for (5, 3) are (0.034598, 0.0089861).

In summary, the first mover calls size 6 alliance. The first two alliances' payoffs from (6, 4) are (0.022558, 0.0081571).

4.2 Case 2: $\sigma = \frac{5}{6}$ or $\tau = 3$

When $\sigma = \frac{5}{6}$, the general pattern is similar to the case of $\sigma = \frac{6}{7}$, except for one important difference: the case where the first mover calls a size 4 alliance. In this case, if the second mover calls a size 5 alliance, then their payoffs from (4, 5) are (0.017665, 0.024663), leaving the leftover player inactive. Thus, it now pays for the first mover to call a size 4 alliance, since the payoffs for (4, 4) when leaving the rest inactive are (0.027344, 0.027344), which is strictly better than calling a size 5 alliance for the second mover. The key difference between this case and case 1 is the difference in the winning probabilities under (4, 4) and (4, 5). If (4, 4), the winning probabilities are $\frac{1}{2}$ and $\frac{1}{2}$, obviously. In contrast, if (4, 5), the winning probabilities are (0.29058, 0.70942) when $\sigma = \frac{6}{7}$, and (0.033862, 0.66138) when $\sigma = \frac{5}{6}$. That is, when σ is not high enough, it does not pay for the second mover to expand the size of alliance. Proposition 1 says that in stage 3, the subsequent payoffs of the winning alliance is inversely proportional to n^2 , so the winning probability of a larger alliance needs to be significantly higher to be profitable to form it. An increase in τ (σ) increases the winning probability of a larger alliance. That is, Lemma 5 is violated for $\sigma = \frac{5}{6}$, while it is satisfied for $\sigma = \frac{6}{7}$. As a result, in this case, the equilibrium outcome is an (active) alliance structure (4, 4) with payoffs 0.027344.

4.3 Case 3: Smaller σ s

When $\sigma = \frac{4}{5}$ ($\tau = 2$), the situation is the same as in the $\sigma = \frac{5}{6}$ case. The equilibrium (active) alliance structure for this case is (4, 4). How about for an even smaller σ ? When $\sigma = \frac{3}{4}$ ($\tau = 1$), we have an (active) equilibrium alliance structure (3, 3, 3), achieving payoffs 0.028807. Note that this number is higher than the payoff from (4, 4), 0.027344. With this low complementarity, even if the first mover calls a size 3 alliance, the second mover does not benefit by calling a size 4 or 5 alliance. Having a large alliance just intensifies the subsequent fight, and (3, 3, 3) realizes.

When $\sigma = \frac{2}{3}$ ($\tau = 0$), the equilibrium alliance structure is (2, 2, 2, 2, 2) with payoffs 0.03. There will be no further spinoff for this $N = 10$, since calling a one person alliance increases the number of alliances, which is harmful to the player (an independent player gets $\frac{1}{36} < 0.03$ from (2, 2, 2, 2, 1, 1)). However, if N goes up, all alliances are resolved, going back to the standard Tullock competition.

5 Concluding Remarks

In this paper, we consider an alliance formation game in Tullock contests when efforts by the members of an alliance are complementary to each other. In order to illustrate excludability of alliance memberships, we use Bloch's noncooperative game of sequential coalition formation (1996). Unlike in an open-membership game analyzed in the companion paper (Konishi and Pan 2019), strong complementarity does not mean a grand alliance, since alliances can exclude outsiders by limiting membership. We show that there will be only two asymmetric alliances in which (i) all players belong to one of them, and (ii) the first alliance is larger than the second alliance, when effort complementarity is large enough. With a small population example, we show that (i) there can be more than two alliances in equilibrium, and (ii) there can be fringe inactive players in equilibrium when effort complementarity is not too strong. These results sheds light on the role of exclusivity in forming alliances in the context of contest games.

Appendix

We collect all the proofs of lemmas in the text.

Proof of Lemma 1. We start by differentiating f and g with respect to \bar{x} :

$$\frac{\partial f(x, \bar{x}; J+1)}{\partial \bar{x}} = \frac{-\tau J \frac{1}{\bar{x}^{\tau+1}} J \frac{1}{x^{\tau+1}}}{\left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right)^2} < 0$$

and

$$\begin{aligned} \frac{\partial g(x, \bar{x}; J+1)}{\partial \bar{x}} &= \frac{-\tau J \frac{1}{\bar{x}^{\tau+1}} \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right) + \left(J \frac{1}{\bar{x}^\tau} - (J-1) \frac{1}{x^\tau}\right) \tau J \frac{1}{\bar{x}^{\tau+1}}}{x^2 \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right)^2} \\ &= \frac{-\tau J \frac{1}{\bar{x}^{\tau+1}} \left[\left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right) - \left(J \frac{1}{\bar{x}^\tau} - (J-1) \frac{1}{x^\tau}\right)\right]}{x^2 \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right)^2} \\ &= \frac{-\tau J \frac{1}{\bar{x}^{\tau+1}} \times J \frac{1}{x^\tau}}{x^2 \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right)^2} < 0 \end{aligned}$$

These imply that $\frac{\partial u(x, \bar{x}; J+1)}{\partial \bar{x}} < 0$: i.e., a coalition's payoff declines if other active coalitions' sizes increase.

Differentiating f and g with respect to x , we have

$$\begin{aligned} \frac{\partial f(x, \bar{x}; J+1)}{\partial x} &= \frac{J(\tau+1) \frac{1}{x^{\tau+2}} \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right) + \frac{1}{x^{\tau+1}} \left(-\tau \frac{1}{x^{\tau+1}}\right)}{N \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right)^2} \\ &= \frac{J(\tau+1) J \frac{1}{x^{\tau+2}} \frac{1}{\bar{x}^\tau} + \frac{1}{x^{2\tau+2}}}{N \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right)^2} > 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial g(x, \bar{x}; J+1)}{\partial x} &= \frac{(J-1) \frac{\tau}{x^{\tau+1}} x^2 \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right) - \left(J \frac{1}{\bar{x}^\tau} - (J-1) \frac{1}{x^\tau}\right) \left(2x \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right) - x^2 \frac{\tau}{x^{\tau+1}}\right)}{x^4 \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right)^2} \\ &= \frac{(J-1) \frac{\tau}{x^{\tau+1}} x \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right) - \left(J \frac{1}{\bar{x}^\tau} - (J-1) \frac{1}{x^\tau}\right) \left(2 \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right) - x \frac{\tau}{x^{\tau+1}}\right)}{x^3 \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right)^2} \\ &= \frac{\left[\left(J-1\right) \frac{\tau}{x^\tau} - 2 \left(J \frac{1}{\bar{x}^\tau} - (J-1) \frac{1}{x^\tau}\right)\right] \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right) + \left(J \frac{1}{\bar{x}^\tau} - (J-1) \frac{1}{x^\tau}\right) \frac{\tau}{x^\tau}}{x^3 \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right)^2} \\ &= \frac{\left[\left(J-1\right) (\tau+2) \frac{1}{x^\tau} - 2J \frac{1}{\bar{x}^\tau}\right] \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right) + \left(J \frac{1}{\bar{x}^\tau} - (J-1) \frac{1}{x^\tau}\right) \frac{\tau}{x^\tau}}{x^3 \left(J \frac{1}{\bar{x}^\tau} + \frac{1}{x^\tau}\right)^2} \quad (4) \end{aligned}$$

Thus, $\frac{\partial g(x, \bar{x}; J+1)}{\partial x} > 0$ (thus $\frac{\partial u(x, \bar{x}; J+1)}{\partial x} > 0$) holds if we have

$$\frac{J-1}{J} \leq \left(\frac{x}{\bar{x}}\right)^\tau \leq \frac{(2+\tau)(J-1)}{2J}$$

We can relax the sufficient upperbound slightly:

$$\begin{aligned}
& \text{(numerator of (4))} \\
&= (J-1)(\tau+2)\frac{1}{x^{2\tau}} + \{J(J-1)(\tau+2) - 2J\}\frac{1}{x^\tau \bar{x}^\tau} - 2J^2\frac{1}{\bar{x}^{2\tau}} - (J-1)\frac{\tau}{x^{2\tau}} + J\frac{\tau}{x^\tau \bar{x}^\tau} \\
&= 2(J-1)\frac{1}{x^{2\tau}} + J\{(\tau+2)J - 4\}\frac{1}{x^\tau \bar{x}^\tau} - 2J^2\frac{1}{\bar{x}^{2\tau}} \\
&= 2(J-1)\frac{1}{x^{2\tau}} + \frac{J}{\bar{x}^\tau} \left[\{(\tau+2)J - 4\}\frac{1}{x^\tau} - 2J\frac{1}{\bar{x}^\tau} \right]
\end{aligned}$$

Thus, $\frac{\partial g(x, \bar{x}; J+1)}{\partial x} > 0$ (thus $\frac{\partial u(x, \bar{x}; J+1)}{\partial x} > 0$) holds if we have

$$\left(\frac{x}{\bar{x}}\right)^\tau \leq \frac{(2+\tau)J - 4}{2J}$$

What if $\left(\frac{x}{\bar{x}}\right)^\tau < \frac{J-1}{J}$? Suppose that originally J alliances are active with average alliance size \bar{x} . Then, even if $J+1$ th alliance with size x enters, this alliance cannot be active if $J\frac{1}{\bar{x}^\tau} - (J-1)\frac{1}{x^\tau} < 0$, or

$$\left(\frac{x}{\bar{x}}\right)^\tau < \frac{J-1}{J}.$$

We have completed the proof. ■

Proof of Lemma 2. Suppose that J_1 size $x_1 = \frac{n_1}{N}$ coalitions and J_2 size $x_2 = \frac{n_2}{N}$ coalitions. We show that if $x_1 \leq x_2$, then there is $\bar{\tau}(x_1, x_2, J_1, J_2)$ such that for all $\tau > \bar{\tau}(x_1, x_2, J_1, J_2)$ and all $x \leq x_1 - \frac{1}{N}$,

$$(J_1 + J_2) \frac{1}{\left(x_1 - \frac{1}{N}\right)^\tau} > J_1 \frac{1}{x_1^\tau} + J_2 \frac{1}{x_2^\tau} + \frac{1}{\left(x_1 - \frac{1}{N}\right)^\tau}$$

For N , there are finite numbers of n_1, n_2, J_1 , and J_2 , and there is a maximum threshold value for $\bar{\tau}(x_1, x_2, J_1, J_2)$. Let it be $\bar{\tau}(N)$. We have completed the proof. ■

Proof of Lemma 3. Direct calculations show that the payoffs of size- x_2 alliances under π and π' are:

$$\frac{1}{N^2 x_2^2} \left[1 - \frac{x_2^\tau}{x_1^\tau + x_2^\tau} \right] \left[1 - \frac{1}{N x_2} \frac{x_2^\tau}{x_1^\tau + x_2^\tau} \right]$$

and

$$\frac{1}{N^2 x_2^2} \left[1 - \frac{J x_2^\tau}{x_1^\tau + J x_2^\tau} \right] \left[1 - \frac{1}{N x_2} \frac{J x_2^\tau}{x_1^\tau + J x_2^\tau} \right]$$

respectively. We have

$$\begin{aligned} \frac{Jx_2^\tau}{x_1^\tau + Jx_2^\tau} - \frac{x_2^\tau}{x_1^\tau + x_2^\tau} &= \frac{Jx_2^\tau(x_1^\tau + x_2^\tau) - x_2^\tau(x_1^\tau + Jx_2^\tau)}{(x_1^\tau + Jx_2^\tau)(x_1^\tau + x_2^\tau)} \\ &= \frac{(J-1)x_2^\tau x_1^\tau}{(x_1^\tau + Jx_2^\tau)(x_1^\tau + x_2^\tau)} > 0 \end{aligned}$$

Thus, we have completed the proof. ■

Proof of Lemma 4. Let $\epsilon = \frac{1}{N}$. By Lemma 2, there will be at most two active alliance sizes. Thus, if $x > x^M$ then direct calculations show

$$\begin{aligned} u(x^M + \epsilon, x^M; J^M) &= \frac{1}{N^2(x^M + \epsilon)^2} \left[1 - \frac{\frac{J^M}{(x^M + \epsilon)^\tau}}{\frac{1}{(x^M + \epsilon)^\tau} + \frac{J^M}{(x^M)^\tau}} \right] \left[1 - \frac{\frac{J^M}{N(x^M + \epsilon)^{\tau+1}}}{\frac{1}{(x^M + \epsilon)^\tau} + \frac{J^M}{(x^M)^\tau}} \right] \\ &= \frac{1}{N^2(x^M + \epsilon)^2} \left[1 - \frac{J^M}{1 + J^M \frac{(x^M + \epsilon)^\tau}{(x^M)^\tau}} \right] \left[1 - \frac{1}{N(x^M + \epsilon)} \frac{J^M}{1 + J^M \frac{(x^M + \epsilon)^\tau}{(x^M)^\tau}} \right] \end{aligned}$$

and (since $x < x^M$ is dominated by x^M ; Lemma 1),

$$u(x^M, x^M; J^M) = \frac{1}{N^2(x^M)^2} \frac{1}{J^M + 1} \left[1 - \frac{J^M}{(J^M + 1)N x^M} \right]$$

Note that if $u(x^M + \epsilon, x^M; 1) > u(x^M, x^M; 1)$ for some τ , then $u(x^M + \epsilon, x^M; J) > u(x^M, x^M; J)$ also holds for all $J \geq 2$. If τ is large, $\frac{(x^M + \epsilon)^\tau}{(x^M)^\tau}$ can be made arbitrarily large. Thus, we can show $u(x^M + \epsilon, x^M; J) > u(x^M, x^M; J)$. ■

Proof of Lemma 5. Let $\epsilon = \frac{1}{N}$. We have

$$u(x, x + \epsilon; 1) = \frac{1}{N^2 x^2} \left[1 - \frac{\frac{1}{x^\tau}}{\frac{1}{(x + \epsilon)^\tau} + \frac{1}{x^\tau}} \right] \left[1 - \frac{1}{N(x + \epsilon)} \times \frac{\frac{1}{(x + \epsilon)^\tau}}{\frac{1}{(x + \epsilon)^\tau} + \frac{1}{x^\tau}} \right]$$

and

$$u\left(\frac{1}{2} + \epsilon, \frac{1}{2} - \epsilon; 1\right) = \frac{1}{N^2 \left(\frac{1}{2} + \epsilon\right)^2} \left[1 - \frac{\frac{1}{\left(\frac{1}{2} + \epsilon\right)^\tau}}{\frac{1}{\left(\frac{1}{2} + \epsilon\right)^\tau} + \frac{1}{\left(\frac{1}{2} - \epsilon\right)^\tau}} \right] \left[1 - \frac{1}{N \left(\frac{1}{2} + \epsilon\right)} \times \frac{\frac{1}{\left(\frac{1}{2} + \epsilon\right)^\tau}}{\frac{1}{\left(\frac{1}{2} + \epsilon\right)^\tau} + \frac{1}{\left(\frac{1}{2} - \epsilon\right)^\tau}} \right]$$

Note that $\frac{\frac{1}{x^\tau}}{\frac{1}{(x + \epsilon)^\tau} + \frac{1}{x^\tau}} > \frac{1}{2}$ and $\frac{\frac{1}{\left(\frac{1}{2} + \epsilon\right)^\tau}}{\frac{1}{\left(\frac{1}{2} + \epsilon\right)^\tau} + \frac{1}{\left(\frac{1}{2} - \epsilon\right)^\tau}} < \frac{1}{2}$ monotonically converge to 1 and 0 as τ goes to infinity, respectively. Thus, for all τ large enough,

$$\frac{1}{x^2} \left[1 - \frac{\frac{1}{x^\tau}}{\frac{1}{(x + \epsilon)^\tau} + \frac{1}{x^\tau}} \right] < \frac{1}{\left(\frac{1}{2} + \epsilon\right)^2} \left[1 - \frac{\frac{1}{\left(\frac{1}{2} + \epsilon\right)^\tau}}{\frac{1}{\left(\frac{1}{2} + \epsilon\right)^\tau} + \frac{1}{\left(\frac{1}{2} - \epsilon\right)^\tau}} \right]$$

holds. This completes the proof of the claim. ■

Proof of Lemma 6. We have

$$\begin{aligned}
& u(x^M, x^M + \frac{1}{N}; J^M) \\
&= \frac{1}{N^2 (x^M)^2} \left[1 - \frac{J^M \frac{1}{(x^M)^\tau}}{\frac{1}{(x^M + \frac{1}{N})^\tau} + J^M \frac{1}{(x^M)^\tau}} \right] \left[1 - \frac{1}{N (x^M + \frac{1}{N})} \times \frac{J^M \frac{1}{(x^M)^\tau}}{\frac{1}{(x^M + \frac{1}{N})^\tau} + J^M \frac{1}{(x^M)^\tau}} \right] \\
&= \frac{1}{N^2 (x^M)^2} (1 - A(\tau)) \left(1 - \frac{A(\tau)}{N (x^M + \frac{1}{N})} \right)
\end{aligned}$$

and

$$\begin{aligned}
& u\left(\frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) + \frac{1}{N}, \frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) - \frac{1}{N}; 1\right) \\
&= \frac{1}{N^2 \left(\frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) + \frac{1}{N}\right)^2} \left[1 - \frac{\frac{1}{\left(\frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) + \frac{1}{N}\right)^\tau}}{\frac{1}{\left(\frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) + \frac{1}{N}\right)^\tau} + \frac{1}{\left(\frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) - \frac{1}{N}\right)^\tau}} \right] \\
&\quad \times \left[1 - \frac{1}{N \left(\frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) + \frac{1}{N}\right)} \times \frac{\frac{1}{\left(\frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) + \frac{1}{N}\right)^\tau}}{\frac{1}{\left(\frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) + \frac{1}{N}\right)^\tau} + \frac{1}{\left(\frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) - \frac{1}{N}\right)^\tau}} \right] \\
&= \frac{1}{N^2 \left(\frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) + \frac{1}{N}\right)^2} (1 - B(\tau)) \left(1 - \frac{B(\tau)}{N \left(\frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) + \frac{1}{N}\right)} \right)
\end{aligned}$$

Note that $A(\tau) = \frac{J^M \frac{1}{(x^M)^\tau}}{\frac{1}{(x^M + \frac{1}{N})^\tau} + J^M \frac{1}{(x^M)^\tau}} > \frac{1}{2}$ and $B(\tau) = \frac{\frac{1}{\left(\frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) + \frac{1}{N}\right)^\tau}}{\frac{1}{\left(\frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) + \frac{1}{N}\right)^\tau} + \frac{1}{\left(\frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) - \frac{1}{N}\right)^\tau}} < \frac{1}{2}$ monotonically converge to 1 and 0 as τ goes to infinity, respectively. Thus, for all τ large enough,

$$u(x^M, x^M + \frac{1}{N}; J^M) < u\left(\frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) + \frac{1}{N}, \frac{1}{2} \left(1 - \sum_{j=1}^J x_j\right) - \frac{1}{N}; 1\right)$$

holds. This completes the proof of the lemma. ■

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