Real Analysis Qualifying Exam

Answer all four questions. In your proofs, you may use any major theorem, except the result you are trying to prove (or a variant of it). State clearly what theorems you use. All four questions are worth the same number of points. Good luck.

Question 1.

a. (8 points.) For a finite measure space, prove that $1 \leq q \leq p \leq \infty$ implies $L^q \supseteq L^p$.

b. (2 points.) Consider the unit interval $[0,1]$ with Lebesgue measure. Show, by example, that for each $p \in \mathbb{N}$, there exists $f \in L^p$ such that $f \notin L^{p+1}$.

Question 2. Let X and Y be Banach spaces.

a. (5 points) Prove that the linear space $X \oplus Y$ is a Banach space under the norm $||(x,y)|| = ||x|| + ||y||$.
(State clearly what properties you are proving.)

b. (5 points) Explicitly determine (with justification) the dual $(X \oplus Y)^*$.

Question 3.

a. (3 points) State the Lebesgue-Radon-Nikodym Theorem. Also state what "absolutely continuous" and "singular" mean in this context.

For questions 3.b. and 3.c., let $(X, \mathcal{M})$ denote the unit interval $[0,1]$ equipped with the Borel σ-algebra, let ν be Lebesgue measure on $(X, \mathcal{M})$ and let μ be counting measure on $(X, \mathcal{M})$.

b. (4 points) Either explicitly determine $\frac{d\nu}{d\mu}$ or prove that it does not exist.

c. (3 points) Set $C = \{(x,y) \in X \times X \mid y = x^2\}$. Calculate (with justification)

$$\int\int \chi_C \; d\mu \; d\nu, \quad \int\int \chi_C \; d\nu \; d\mu, \quad \int \chi_C \; d(\mu \times \nu)$$

Question 4. (10 points) Let $f : [0, \infty) \to \mathbb{R}$ be Lebesgue integrable. Prove that if f is uniformly continuous, then $\lim_{x\to\infty} f(x) = 0$.