Algebra Qualifying Exam, Fall 2013
You have 3 hours to answer all problems.

1. Classify, up to isomorphism, all groups of order $385 = 5 \cdot 7 \cdot 11$.

2. Determine the Galois group of the polynomial $X^5 - 2 \in \mathbb{Q}[X]$.

3. Let R be a local ring with maximal ideal $\mathfrak{M}$. Suppose that $f : A \to B$ is a homomorphism of finitely generated free R-modules with the property that the induced map $A/\mathfrak{M}A \to B/\mathfrak{M}B$ is an isomorphism Show that f is itself an isomorphism.

4. The ring of integers of $\mathbb{Q}[\sqrt{7}]$ is $\mathbb{Z}[\sqrt{7}]$. For each of the following primes $p \in \mathbb{Z}$, describe how the ideal $p\mathbb{Z}[\sqrt{7}]$ factors as a product of prime ideals ("describe" means give the number of prime factors, their multiplicities in the factorization, and the cardinalities of the residue fields):

(a) $p = 2$

(b) $p = 7$

(c) $p = 17$.

5. Let A be an $n \times n$ matrix with entries in an algebraically closed field. Show that A is similar to a diagonal matrix if and only if the minimal polynomial of A has no repeated roots.

6. Let R be a commutative ring with 1, N an R-module, and for every maximal ideal $\mathfrak{m} \subset R$ let $N_{\mathfrak{m}}$ be the localization of N at $\mathfrak{m}$. Prove that the natural map $N \to \prod_{\mathfrak{m}} N_{\mathfrak{m}}$ is injective.

7. Let k be a field, $R = k[x, y]$ and $I = (x, y)$.

(a) Prove that I is neither flat nor projective as an R-module.

(b) Compute $\mathrm{Ext}^1_R(R/I, I)$.

8. Let k be an algebraically closed field. Consider the affine variety $V = k^2$ with coordinates x, y, and the affine variety $W = k^2$ with coordinates s, t. Suppose $f \colon V \to W$ a morphism, and denote by R the image of the induced pull-back map $f^* \colon k[s, t] \to k[x, y]$. For each of the following statements, give a proof or a counterexample.

(a) If f has Zariski dense image, then f is surjective.

(b) If $k[x, y]/R$ is an integral extension of rings, then f is surjective.