Algebra Qualifying Exam
Spring 2014

1. Let $\mathbb{F}_2$ be the field with 2 elements.

(a) Determine the Galois group of $x^5 + x^3 + x^2 + x + 1 \in \mathbb{F}_2[x]$.

(b) Exhibit a matrix of order 31 in $\mathrm{GL}_5(\mathbb{F}_2)$.

2. Suppose $p > 2$ is prime. Classify the groups of order $2p^2$.

3. Let F be a field and suppose the minimal polynomial of $A \in M_n(F)$ has degree n. Show that every matrix commuting with A has the form $p(A)$ for a polynomial $p(x) \in F[x]$.

4. Suppose R is a Noetherian local ring and M is a finitely generated flat R-module. Prove that M is free.

5. Fix a finite extension K/F of subfields of $\mathbb{C}$, and $\zeta \in \mathbb{C}$.

(a) If ζ is transcendental over F, prove that $[K(\zeta) : F(\zeta)] = [K : F]$.

(b) Find and example of F, K, and <u>algebraic</u> ζ such that $[K(\zeta) : F(\zeta)]$ is not a divisor of $[K : F]$.

6. Define $R = \mathbb{C}[x, y]/(y^4 + x^2 - 1)$.

(a) Show that R is an integral domain.

(b) Let K be the fraction field of R. Show that K is Galois over $\mathbb{C}(x)$, and compute the Galois group.

(c) For each prime ideal of $\mathbb{C}[x]$, determine the number of primes of R lying above it, and find generators for those primes.

7. Let $R = k[x]$ and $M = k[x, y]/(xy)$. Show that each of the following R-modules is isomorphic to a direct sum of cyclic factors, and describe the factors.

(a) $\mathrm{Tor}_1^R(M, R/(x))$.

(b) $\mathrm{Ext}_R^1(R/(x), M)$.

8. Let k be a field. Find the Krull dimensions of

$$R = k[x, y, z]/(xz, yz),$$

$R/(x + y)$, and $R/(x + y + z)$.