Algebra Qualifying Exam
Spring 2015
3 hours

1. (a) Show that $\mathrm{GL}_2(\mathbb{F}_5)$ has a unique conjugacy class of elements of order three.

(b) Classify, up to isomorphism, all groups of order $3 \cdot 5^2$, and give a presentation for each group. Hint: $\mathrm{Aut}(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}) \cong \mathrm{GL}_2(\mathbb{F}_5)$.

2. Suppose F is a field and $a \in F$. For each of the following groups G, either find an example of F and a for which $x^4 - a \in F[x]$ has Galois group G, or show that no such F and a exist.

$$G = D_8, \qquad G = S_4, \qquad G = \mathbb{Z}/4\mathbb{Z}.$$

3. Suppose p is a prime. Show that the Galois group of $x^5 - 1 \in \mathbb{F}_p[x]$ depends only on $p \pmod 5$, and compute it for each congruence class $p \pmod 5$.

4. Suppose R is a Noetherian local ring with maximal ideal $\mathfrak{m}$. If $\mathfrak{a}$ is an ideal such that the *only* prime ideal containing $\mathfrak{a}$ is $\mathfrak{m}$, show that $\mathfrak{m}^k \subset \mathfrak{a}$ for $k \gg 0$.

5. Suppose R is a UFD, and let $R_{\mathfrak{p}}$ be the localization of R at a prime $\mathfrak{p} = (\pi)$ generated by an irreducible element. Prove that every ideal of $R_{\mathfrak{p}}$ is principal.

6. Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$, and let M be a finitely generated R-module. Show that if $\mathrm{Tor}_1^R(M, k) = 0$ then M is a free R-module.

7. Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be an ideal such that the zero locus

$$V(I) = \{x \in \mathbb{C}^n : f(x) = 0, \forall f \in I\}$$

is finite.

(a) If $I = \sqrt{I}$ is a radical ideal, show that the quotient $\mathbb{C}[x_1, \ldots, x_n]/I$ has dimension $\#V(I)$ as a $\mathbb{C}$-vector space.

(b) Without assuming that I is radical, show that $\mathbb{C}[x_1, \ldots, x_n]/I$ is finite dimensional. Hint: set $J = \sqrt{I}$ and prove that each J^k/J^{k+1} a finite dimensional $\mathbb{C}$-vector space.

8. Let k be an algebraically closed field. Let V be the algebraic subset of $\mathbb{A}^2$ over k cut out by the equation $y^2 = x^3 + x^2$.

(a) Show that the normalization of $k[V]$ is the polynomial ring $k[t]$ where $t = y/x$.

(b) Compute the fibers of the map $\varphi : \mathbb{A}^1 \to V$ that corresponds to the inclusion $\varphi^* : k[V] \hookrightarrow k[t]$.