ANALYSIS QUALIFYING EXAM

JUNE 2013

Part 1: Real analysis

Answer all 4 questions. In your proofs, you may use any major theorem, except the fact you are trying to prove (or a variant of it). State clearly what theorems you use. Good luck.

Exercise 1. (25 points).

(1) Prove that not every subset of $[0,1]$ is Lebesgue measurable
(2) Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of Lebesgue measurable functions. Prove that the set $E = \{x | \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is Lebesgue measurable

Exercise 2. (25 points).
Given $f \in L^1(\mathbb{R})$ let

$$Hf(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(x)|dx,$$

denote the Hardy-Littlewood maximal function. Show that Hf is measurable and that

$$m(\{x|Hf(x) > \alpha\}) \leq \frac{3}{\alpha} \|f\|_1 .$$

Exercise 3. (25 points).
Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L^\infty(X) \cap L^1(X)$. Show that $f \in L^p(X)$ for all $p > 1$ and that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

Exercise 4. (25 points).
Let X be a Banach space and $L(X,X)$ the space of bounded linear operators.

(1) Show that the space $L(X,X)$ with the induced operator norm is also a Banach space.
(2) Show that the subset $\{T \in L(X,X) | T \text{ is invertible and } T^{-1} \in L(X,X)\}$ is open.

Part 2: Complex analysis

In all of the following, you may freely use the Cauchy integral formula and Cauchy estimates, and the fact that holomorphic functions are analytic. Each problem is worth 20 points.

1. Let $f : \mathbf{C} \to \mathbf{C}$ be an entire holomorphic function such that $|f(z)| \leq \log(|z|)$ whenever $|z|$ is sufficiently large. Show that f is constant.

2. Let $f : \{z \in \mathbf{C} \mid |z| < 1\} \to \mathbf{C}$ be a holomorphic function on the unit disk. Suppose that $f(a_n) = 0$ for some nonzero sequence a_n converging to 0. Show that f is identically zero. Show that this need not be true if a_n converges to 1.

3. Give an example of a nonzero harmonic function $f : \mathbf{C} \to \mathbf{R}$ and a nonzero sequence a_n converging to zero, such that $f(a_n) = 0$ for all n.

4. Let $f(z) = z^4 - 1$. Describe

$$\int_{-\infty}^{\infty} \frac{1}{f(x + ir)} dx$$

as a function of $r \in \mathbf{R}$.

5. Suppose that $f(z) : \mathbf{C} \to \mathbf{C}$ is an entire holomorphic function without any zeroes. Show that there exists a holomorphic $g(z)$ such that $f(z) = e^{g(z)}$, by giving an integral formula for $g(z)$.