# Real Analysis Qual

June 10, 2016

**Problem 1.** Let $\sim$ be the equivalence relation on the interval $[0,1]$ given by $x \sim y$ iff $x - y \in \mathbb{Q}$. Choose one element from each equivalence class (using the Axiom of Choice). Let $A \subset [0,1]$ denote the set of these chosen elements. For a given set $B \subset \mathbb{R}$, define $B + x = \{y + x \mid y \in B\}$.

(a) Show that the sets $A + q$, for $q \in \mathbb{Q} \cap [-1,1]$, are disjoint.

(b) Show that $[0,1] \subset \bigcup_{q \in \mathbb{Q} \cap [-1,1]} (A + q) \subset [-1,2]$.

(c) Show that $A$ is not Lebesgue measurable (supposing it were, what would its measure be?).

**Problem 2.** Suppose $(X, \mathcal{A}, \mu)$ is a measure space.

(a) What does it mean for a function $f : X \to \mathbb{R}$ to be measurable?

(b) State the Monotone Convergence Theorem for this measure space.

(c) Prove Fatou's Lemma, which states that

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$$

for any sequence $(f_n)$ of non-negative, measurable functions. To get started, let $g_n = \inf_{i \geq n} f_i$. Observe these are increasing. What is their limit? Observe that

$$\int g_n \leq \inf_{i \geq n} \int f_i$$

(why?). Finish the proof.

**Problem 3.** Suppose $\mu$ is a positive measure on $(X, \mathcal{A})$, with $\mu(X) < \infty$. Fix $E_1, \ldots, E_n \in \mathcal{A}$ and $c_1, \ldots, c_n \in \mathbb{R}_{\geq 0}$. Define $\nu : \mathcal{A} \to [0, \infty]$ by

$$\nu(A) = \sum_{i=1}^{n} c_i \mu(A \cap E_i).$$

(a) Show that $\nu$ is a measure.

(b) Show that $\nu$ is absolutely continuous with respect to $\mu$.

(c) State the Radon-Nikodym theorem with respect to $\nu$ and $\mu$.

(d) Find the Radon-Nikodym derivative $f = \frac{d\nu}{d\mu}$.

**Problem 4.**

(a) Define the $L^p$ norm on the measure space $(X, \mathcal{A}, \mu)$.

(b) What does it mean for $H : L^p(X, \mathcal{A}, \mu) \to \mathbb{R}$ to be a bounded linear functional?

(c) Prove that a bounded linear functional is continuous with respect to the metric topology on $L^p(X, \mathcal{A}, \mu)$ induced by the $L^p$ norm.

# Complex Analysis Qualifying Exam – Spring 2016

Please answer the following problems. Explain your argument carefully – if you refer to a well-known theorem from class, please state the theorem precisely and explain why it applies.
Notation: $\mathbb{D}$ = open unit disk, $\mathbb{C}^* = \mathbb{C} - \{0\}$, $\mathbb{H}$ = upper half plane.

1) Find a holomorphic function $f$ on $\mathbb{C}^*$ such that

   - $f$ is a pointwise limit of polynomial functions, but
   - $f$ is not a uniform limit of polynomial functions (that is, there is no sequence of polynomials that converges to $f$ uniformly on compact subsets of $\mathbb{C}^*$).

   Prove both assertions for your choice of $f$.

2) Find a biholomorphism between $\mathbb{H}$ and the region

$$U = \{ z \in \mathbb{C} \, | \, |z-1| < 1, |z-i| < 1 \} .$$

   It is enough to write down explicitly functions whose composition yields a biholomorphism from $\mathbb{H}$ to $U$ or from $U$ to $\mathbb{H}$.

3) Let $U \subsetneq \mathbb{C}$ be a simply connected region. For any point $a \in U$, the Green function of $U$ corresponding to $a$ is defined as $g_a(z) = \log|\phi(z)|$, where $\phi : U \to \mathbb{D}$ is a biholomorphism sending $a$ to 0.

   a) Explain why $g_a$ is well-defined. That is, explain briefly why such a map $\phi$ is guaranteed to exist, and then show carefully that $g_a$ is independent of the choice of $\phi$.
   b) Prove that for any $a, b \in U$ we have $g_a(b) = g_b(a)$.

4) Fix a lattice $\Lambda$ and define the Weierstrass $\sigma$-function:

$$\sigma(z) = z \prod_{\omega \in \Lambda - \{0\}} \left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2}\right)$$

   a) Verify that the product converges to an odd holomorphic function with a simple zero exactly at every lattice point.
   b) Show that

$$\frac{d}{dz} \frac{\sigma'(z)}{\sigma(z)} = -\wp(z)$$

where $\wp(z)$ is the Weierstrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \left( \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

Justify carefully any techniques you use.